The tacnode kernel: equality of Riemann-Hilbert and Airy resolvent formulas

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Abstract

We study nonintersecting Brownian motions with two prescribed starting and ending positions, in the neighborhood of a tacnode in the time-space plane. Several expressions have been obtained in the literature for the critical correlation kernel $K_{\mathrm{tac}}(x,y)$ that describes the microscopic behavior of the Brownian motions near the tacnode. One approach, due to Kuijlaars, Zhang and the author, expresses the kernel (in the single time case) in terms of a 4×4 matrix valued Riemann-Hilbert problem. Another approach, due to Adler, Ferrari, Johansson, van Moerbeke and Vető in a series of papers, expresses the kernel in terms of resolvents and Fredholm determinants of the Airy integral operator acting on a semi-infinite interval $[\sigma,\infty)$, involving some objects introduced by Tracy and Widom. In this paper we prove the equivalence of both approaches. We also obtain a remarkable rank-2 property for the derivative of the tacnode kernel.

Keywords: resolvent operator, Fredholm determinant, Tracy-Widom distribution, Riemann-Hilbert problem, Lax pair, Painlevé II equation, Brownian motion, determinantal point process.

1 Introduction

Recently several papers appeared that study n non-intersecting Brownian motion paths with prescribed starting positions at time $t=0$ and ending positions at time $t=1$, see [1, 2, 4, 12, 13, 15, 16, 21, 22, 23] among many others. If $n\to\infty$ then the paths fill up a well-defined region in the time-space plane. By fine-tuning the parameters, we may create a situation with two groups of Brownian motions, located inside two touching ellipses in the time-space plane: see the third picture of Figure 1. We are interested in the microscopic behavior of the paths near the touching point of the two ellipses, i.e., near the tacnode.

It is well-known that the positions of the Brownian motions at any fixed time $t\in(0,1)$ form a determinantal point process. The process has a well-defined limit for $n\to\infty$ in a microscopic neighborhood of the tacnode. The limiting process is encoded by a two-variable correlation kernel $K_{\mathrm{tac}}(x,y)$ which we call the *tacnode kernel.* It depends parametrically on the scaling that we use near the tacnode. There exists also a multi-time extended version of the tacnode kernel [2, 3, 16, 21] but we will not consider it in this paper.

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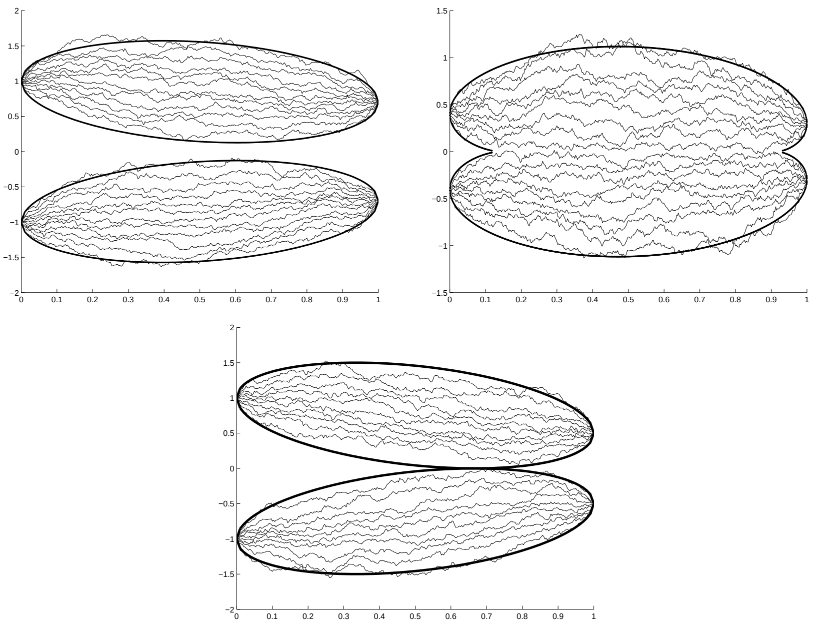

Figure 1: $n = 20$ non-intersecting Brownian motions at temperature $T = 1$ with two prescribed starting and two ending positions in the case of (a) large, (b) small, and (c) critical separation between the endpoints. The horizontal axis stands for the time, $t \in [0, 1]$, and the vertical axis shows the positions of the Brownian motions at time t. For $n \to \infty$ the Brownian motions fill a prescribed region in the time-space plane which is bounded by the boldface lines in the figure. In the case (c), the limiting support consists of two touching ellipses which touch each other at a critical point which is a tacnode.

The tacnode kernel $K_{\mathrm{tac}}(x, y)$ can be expressed using resolvents and Fredholm determinants of the Airy integral operator acting on a semi-infinite interval $[\sigma, \infty)$. This approach was followed in the symmetric case by Adler, Ferrari, van Moerbeke [2] and Johansson [21] and in the non-symmetric case by Ferrari and Vető [16]. Here the 'symmetric case' means that the two touching groups of Brownian motions have the same size, as in Figure 1, and the non-symmetric case means that one group is bigger than the other. Similar methods were used to study a double Aztec diamond [3].

An alternative expression for the tacnode kernel can be obtained from the Riemann-Hilbert method. This approach was followed by Kuijlaars, Zhang and the author in [13]. In that paper we express the tacnode kernel in terms of a 4×4 matrix-valued Riemann-Hilbert problem (RH problem) $M(z)$, which yields a new Lax pair representation for the Hastings-McLeod solution $q(x)$ to the Painlevé II equation. Recall that the *Painlevé II equation* is the second-order, ordinary differential equation

$$q''(x) = xq(x) + 2q^3(x), \tag{1.1}$$

where the prime denotes the derivative with respect to x. The *Hastings-McLeod solution* [18, 20] is the special solution $q(x)$ of (1.1) that is real for real x and satisfies

$$q(x) \sim \mathrm{Ai}(x), \qquad x \to +\infty, \tag{1.2}$$

with Ai the Airy function. We note that the usual Riemann-Hilbert matrix $\Psi(z)$ associated to the Painlevé II equation, due to Flaschka and Newell [17], has size 2×2 rather than 4×4.

The RH matrix $M(z)$ from the previous paragraph has been the topic of some recent developments. It was used to study a new critical phenomenon in the two-matrix model [14], and to establish a reduction from the tacnode kernel to the Pearcey kernel [19]. It was also extended to a hard-edge version of the tacnode [11].

Summarizing, there exist several, apparently different, formulas for the tacnode kernel $K_{\mathrm{tac}}(x, y)$. It is natural to ask about the equivalence of these formulas.

There is an interesting analogy with a model of non-intersecting Brownian excursions, also known as watermelons (with a wall). The model consists of n Brownian motion paths on the positive half-line with a reflecting or absorbing wall at the origin. The paths are forced to start and end at the origin and the interest lies in the maximum position $x_{\max}$ reached by the topmost path during the time interval $t \in [0, 1]$.

Recently, several results were obtained about the joint distribution of the maximum position $x_{\max}$ and the maximizing time $t_{\max}$ for such watermelons. One approach, due to Moreno-Quastel-Remenik [24], involves resolvents and Fredholm determinants of the Airy operator acting on an interval $[\sigma, \infty)$. Another approach, due to Schehr [25], involves the 2×2 Riemann-Hilbert matrix $\Psi(z)$ associated to the Hastings-McLeod solution to the Painlevé II equation [17, 20]. The equivalence of both approaches has been established in a recent work of Baik-Liechty-Schehr [7]. Along the way they obtain Airy resolvent formulas for the entries of the RH matrix $\Psi(z)$, see also [6, Sec. 1.1.3] for a similar result.

Inspired by the work of Baik-Liechty-Schehr [7], in this paper we obtain Airy resolvent formulas for the 4×4 RH matrix $M(z)$. This is the content of Theorem 2.7 below. In Theorem 2.6 we will use these formulas to prove the equivalence of the tacnode kernels in [2, 3, 16, 21] and [13] respectively. We also obtain a remarkable rank-2 property for the derivative of the tacnode kernel, see Theorems 2.3 and 2.5.

The Airy resolvent formulas will describe the entries in the first and second column of the RH matrix $M(z)$, where z lies in a sector around the positive imaginary axis. We were unable to obtain similar formulas for the third and fourth column of $M(z)$. The latter columns are relevant because they appear in a critical correlation kernel for the 2-matrix model [14]. It is an open problem to obtain Airy type formulas in that case.

Another open problem is to extend our results to the hard-edge situation discussed in [11], and to extend the RH approach to the setting of the multi-time extended tacnode kernel. See [8] for some recent developments that could be useful in the latter direction.

2 Statement of results

2.1 The Riemann-Hilbert problem for $M(z)$

In this section we recall the RH problem for $M(z)$ from [13, 14]. Fix two numbers φ_1, φ_2 such that

$$0 < \varphi_1 < \varphi_2 < \pi/3. \tag{2.1}$$

Define the half-lines Γ_k, $k = 0, \ldots, 9$, by

$$\Gamma_0 = \mathbb{R}_+, \quad \Gamma_1 = e^{i\varphi_1}\mathbb{R}_+, \quad \Gamma_2 = e^{i\varphi_2}\mathbb{R}_+, \quad \Gamma_3 = e^{i(\pi-\varphi_2)}\mathbb{R}_+, \quad \Gamma_4 = e^{i(\pi-\varphi_1)}\mathbb{R}_+, \tag{2.2}$$

and

$$\Gamma_{5+k} = -\Gamma_k, \qquad k = 0, \ldots, 4. \tag{2.3}$$

All rays Γ_k, $k = 0, \ldots, 9$, are oriented towards infinity, as shown in Figure 2. We denote by Ω_k the region in $\mathbb{C}$ that lies between the rays Γ_k and Γ_{k+1}, for $k = 0, \ldots, 9$, where we identify $\Gamma_{10} := \Gamma_0$.

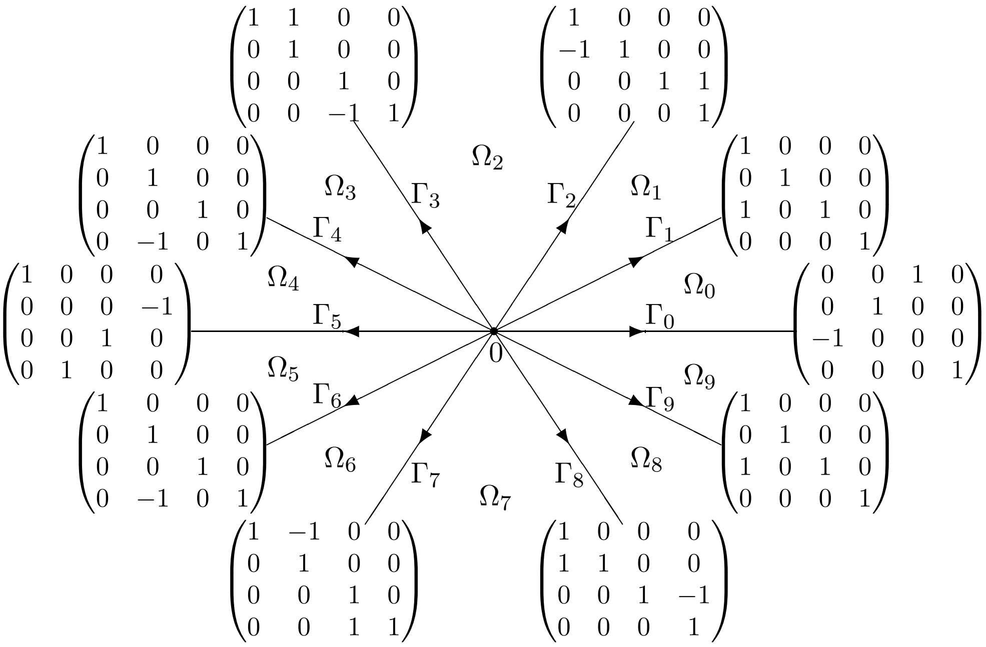

Figure 2: The figure shows the jump contours Γ_k in the complex z-plane and the corresponding jump matrix J_k on Γ_k, $k = 0, \ldots, 9$, in the RH problem for $M = M(z)$. We denote by Ω_k the region between the rays Γ_k and Γ_{k+1}.

We consider the following RH problem.

RH problem 2.1. *We look for a matrix valued function* $M : \mathbb{C} \setminus \left(\bigcup_{k=0}^{9} \Gamma_k \right) \to \mathbb{C}^{4\times 4}$ *(which also depends on the parameters* $r_1, r_2 > 0$ *and* $s_1, s_2, \tau \in \mathbb{C}$*) satisfying*

(1) $M(z)$ *is analytic (entrywise) for* $z \in \mathbb{C} \setminus \left(\bigcup_{k=0}^{9} \Gamma_k \right)$.

(2) For $z \in \Gamma_k$*, the limiting values*

$$M_+(z) = \lim_{z \to z,\, z \text{ on } +\text{-side of } \Gamma_k} M(z), \qquad M_-(z) = \lim_{z \to z,\, z \text{ on } -\text{-side of } \Gamma_k} M(z)$$

exist, where the $+$*-side and* $-$*-side of* Γ_k *are the sides which lie on the left and right of* Γ_k*, respectively, when traversing* Γ_k *according to its orientation. These limiting values satisfy the jump relation*

$$M_+(z) = M_-(z) J_k(z), \qquad k = 0, \ldots, 9, \tag{2.4}$$

where the jump matrix $J_k(z)$ *for each ray* Γ_k *is shown in Figure 2.*

(3) As $z \to \infty$ *we have*

$$M(z) = \left(I + \frac{M_1}{z} + \frac{M_2}{z^2} + O\left(\frac{1}{z^3}\right) \right) \operatorname{diag}((-z)^{-1/4}, z^{-1/4}, (-z)^{1/4}, z^{1/4}) \times \mathcal{A} \operatorname{diag}\left(e^{\theta_1(z)}, e^{\theta_2(z)}, e^{\theta_3(z)}, e^{\theta_4(z)} \right), \tag{2.5}$$

where the coefficient matrices $M_1, M_2, \ldots$ are independent of z, and with

$$\mathcal{A} := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & -i & 0 \\ 0 & 1 & 0 & i \\ -i & 0 & 1 & 0 \\ 0 & i & 0 & 1 \end{pmatrix}, \tag{2.6}$$

and

$$\begin{cases} \theta_1(z) = -\frac{2}{3} r_1 (-z)^{3/2} - 2 s_1 (-z)^{1/2} + r_1^2 \tau z, \\ \theta_2(z) = -\frac{2}{3} r_2 z^{3/2} - 2 s_2 z^{1/2} - r_2^2 \tau z, \\ \theta_3(z) = \frac{2}{3} r_1 (-z)^{3/2} + 2 s_1 (-z)^{1/2} + r_1^2 \tau z, \\ \theta_4(z) = \frac{2}{3} r_2 z^{3/2} + 2 s_2 z^{1/2} - r_2^2 \tau z. \end{cases} \tag{2.7}$$

Here we use the principal branches of the fractional powers.

(4) $M(z)$ is bounded as $z \to 0$.

We will sometimes write $M(z) = M(z; r_1, r_2, s_1, s_2, \tau)$ to indicate the dependence on the parameters. We could assume $r_2 = 1$ without loss of generality by a simple rescaling of z.

The RH problem 2.1 was introduced in [13] with $\tau = 0$ in (2.7). The parameter τ was introduced in [14] in the symmetric setting where $r_1 = r_2 = 1$ and $s_1 = s_2$. The general non-symmetric case with the extra parameter τ has not been considered before in the literature. The following result can be proved as in [14]; see also the discussion following Lemma 6.7 below.

Proposition 2.2. *(Solvability.) For any $r_1, r_2 > 0$ and $s_1, s_2, \tau \in \mathbb{R}$ there is a unique solution $M(z) = M(z; r_1, r_2, s_1, s_2, \tau)$ to the RH problem 2.1.*

2.2 The tacnode kernel and its derivative

Let $r_1, r_2 > 0$ and $s_1, s_2, \tau \in \mathbb{R}$ be fixed parameters. Let $\widehat{M}(z)$ be the restriction of $M(z)$ to the sector $z \in \Omega_2$ around the positive imaginary axis. We extend $\widehat{M}(z)$ to the whole complex z-plane by analytic continuation. This analytic continuation is well-defined, since the product $J_3 J_4 \cdots J_9 J_0 J_1 J_2$ of the jump matrices in the RH problem 2.1 is the identity matrix.

The tacnode kernel is defined in terms of the RH matrix $\widehat{M}(z)$ by [13, Def. 2.6]*

$$K_{\mathrm{tac}}(u, v) = \frac{1}{2\pi i (u - v)} \begin{pmatrix} 0 & 0 & 1 & 1 \end{pmatrix} \widehat{M}^{-1}(v) \widehat{M}(u) \begin{pmatrix} 1 & 1 & 0 & 0 \end{pmatrix}^T, \tag{2.8}$$

where the superscript ^{T} denotes the transpose. It will be convenient to denote by

$$\mathbf{p}(z) = \widehat{M}(z) \begin{pmatrix} 1 & 1 & 0 & 0 \end{pmatrix}^T \in \mathbb{C}^{4 \times 1} \tag{2.9}$$

the sum of the first and second column of $\widehat{M}(z)$. Observe that (2.8)–(2.9) both depend on the parameters r_1, r_2, s_1, s_2, τ.

Note that the kernel (2.8) has an 'integrable' form, due to the factor $u - v$ in the denominator. Interestingly, this factor cancels when taking the derivative with respect to s_1 or s_2. This is the content of the next theorem.

*We note that [13, Def. 2.6] has a typo: it has '$\widehat{M}^{-1}(u)\widehat{M}(v)$' instead of '$\widehat{M}^{-1}(v)\widehat{M}(u)$'.

To state the theorem we parameterize

$$s_1 =: \sigma_1 s, \qquad s_2 =: \sigma_2 s, \tag{2.10}$$

where σ_1, σ_2 are fixed and s is variable. In the symmetric case where $s_1 = s_2 =: s$, we could simply take $\sigma_1 = \sigma_2 = 1$. We also consider $r_1, r_2 > 0$ to be fixed. Then we write $K_{\mathrm{tac}}(u, v; s, \tau)$, $\mathbf{p}(z; s, \tau)$ etc., to denote the dependence on the two parameters s and τ.

Theorem 2.3. *(Derivative of the tacnode kernel.) With the parametrization* (2.10)*, the kernel* (2.8) *satisfies*

$$\frac{\partial}{\partial s} K_{\mathrm{tac}}(u, v; s, \tau) = -\frac{1}{\pi} \left(\sigma_1 p_1(u; s, \tau) p_1(v; s, -\tau) + \sigma_2 p_2(u; s, \tau) p_2(v; s, -\tau) \right), \tag{2.11}$$

where p_j, $j = 1, \ldots, 4$, denotes the jth entry of the vector $\mathbf{p}$ in (2.9)*. Consequently, if $\sigma_1, \sigma_2 > 0$ then*

$$K_{\mathrm{tac}}(u, v; s, \tau) = \frac{1}{\pi} \int_s^\infty \left(\sigma_1 p_1(u; \widetilde{s}, \tau) p_1(v; \widetilde{s}, -\tau) + \sigma_2 p_2(u; \widetilde{s}, \tau) p_2(v; \widetilde{s}, -\tau) \right) \mathrm{d}\widetilde{s}. \tag{2.12}$$

Theorem 2.3 is proved in Section 3. Later we will apply the theorem with the parameters σ_1, σ_2 in (4.39).

Note that the right hand side of (2.11) is a rank-2 kernel. We will see in Section 2.4 below that a similar property holds when starting with the Airy type expressions for the tacnode kernel due to Adler et al. [2, 3, 16, 21].

We point out that (2.11)–(2.12) have an analogue for the kernel $K_\Psi(u, v)$ which is associated to the 2×2 Flaschka-Newell RH matrix $\Psi(z)$, see [10, Eq. (1.22)]. The kernel $K_\Psi(u, v)$ occurs in Hermitian random matrix theory when the limiting eigenvalue density vanishes quadratically at an interior point of its support [9, 10].

2.3 Airy resolvents and the tacnode kernel

Denote by $\mathrm{Ai}(x)$ the standard Airy function and by

$$K_{\mathrm{Ai}}(x, y) = \frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\mathrm{Ai}(y)}{x - y} = \int_0^\infty \mathrm{Ai}(x + z)\mathrm{Ai}(y + z)\, \mathrm{d}z \tag{2.13}$$

the Airy kernel. For $\sigma \in \mathbb{R}$ let

$$K_{\mathrm{Ai},\sigma}(x, y) = \int_0^\infty \mathrm{Ai}(x + z + \sigma)\mathrm{Ai}(y + z + \sigma)\, \mathrm{d}z \tag{2.14}$$

be the Airy kernel shifted by σ. Let $\mathbf{K}_{\mathrm{Ai},\sigma}$ be the integral operator with kernel $K_{\mathrm{Ai},\sigma}$ acting on the function space $L^2([0, \infty))$. The action of the operator $\mathbf{K}_{\mathrm{Ai},\sigma}$ on the function f is defined by

$$\left[\mathbf{K}_{\mathrm{Ai},\sigma} f\right](x) = \int_0^\infty K_{\mathrm{Ai},\sigma}(x, y) f(y)\, \mathrm{d}y.$$

Define the resolvent operator $\mathbf{R}_{\mathrm{Ai},\sigma}$ on $L^2([0, \infty))$ by

$$\mathbf{R}_{\mathrm{Ai},\sigma} := (\mathbf{1} - \mathbf{K}_{\mathrm{Ai},\sigma})^{-1} - \mathbf{1} = \mathbf{K}_{\mathrm{Ai},\sigma}(\mathbf{1} - \mathbf{K}_{\mathrm{Ai},\sigma})^{-1} = (\mathbf{1} - \mathbf{K}_{\mathrm{Ai},\sigma})^{-1}\mathbf{K}_{\mathrm{Ai},\sigma}, \tag{2.15}$$

where $\mathbf{1}$ stands for the identity operator on $L^2([0, \infty))$. It is known that $\mathbf{R}_{\mathrm{Ai},\sigma}$ is again an integral operator on $L^2([0, \infty))$ and we denote its kernel by $R_\sigma(x, y)$:

$$\left[\mathbf{R}_{\mathrm{Ai},\sigma} f\right](x) = \int_0^\infty R_\sigma(x, y) f(y)\, \mathrm{d}y. \tag{2.16}$$

We will sometimes use the notation

$$(\mathbf{1}-\mathbf{K}_{\mathrm{Ai},\sigma})^{-1}(x,y) \equiv (\mathbf{1}+\mathbf{R}_{\mathrm{Ai},\sigma})(x,y) := \delta(x-y)+R_\sigma(x,y), \tag{2.17}$$

with $\delta(x-y)$ the Dirac delta function at $x=y$ and R_σ the Airy resolvent kernel (2.16). We will often use the symmetry of the kernel, $R_\sigma(x,y)=R_\sigma(y,x)$. Finally, we will abbreviate R_σ by R if the value of σ is clear from the context.

In a series of papers [2, 3, 16, 21], Adler, Ferrari, Johansson, van Moerbeke and Vető study the tacnode problem using Airy resolvent expressions. We focus in particular on the paper by Ferrari-Vető [16] on the non-symmetric tacnode. Hence the two touching groups of Brownian motions at the tacnode are allowed to have a different size. The paper [16] uses a parameter $\lambda>0$ that quantifies the amount of asymmetry, with $\lambda=1$ corresponding to the symmetric case treated in [2, 3, 21]. These papers also use a parameter $\sigma>0$ that controls the strength of interaction between the two groups of Brownian motions near the tacnode. In the present paper we will denote the latter parameter with a capital Σ. The parameter Σ has a similar effect on the tacnode kernel as the temperature parameter used in [11, 12] (suitably rescaled). In order to be consistent with [13], we will use the notation σ to denote

$$\sigma=\lambda^{1/2}(1+\lambda^{-1/2})^{2/3}\Sigma. \tag{2.18}$$

(What we call σ was called $\widetilde{\sigma}$ in [2, 3, 16, 21].) The papers [2, 3, 16, 21] consider a multi-time extended tacnode kernel with time variables τ_1,τ_2. We will restrict ourselves to the single time case $\tau_1=\tau_2=:\tau$.

With the above notations, define the functions [16]†

$$\begin{aligned} b_{\tau,z}(x) &= \exp\left(-\tau y+\tau^3/3\right)\mathrm{Ai}(y), && \text{with } y:=z+Cx+\Sigma+\tau^2,\\ \widetilde{b}_{\tau,z}(x) &= \exp\left(-\sqrt{\lambda}\tau\widetilde{y}+\lambda\tau^3/3\right)\mathrm{Ai}(\lambda^{1/6}\widetilde{y}), && \text{with } \widetilde{y}:=-z+Cx+\sqrt{\lambda}(\Sigma+\tau^2), \end{aligned} \tag{2.19}$$

where

$$C=(1+\lambda^{-1/2})^{1/3}. \tag{2.20}$$

Note that in the symmetric case $\lambda=1$, we have $\widetilde{b}_{\tau,z}(x)=b_{\tau,-z}(x)$. The functions (2.19) also depend on λ,σ (recall (2.18)) but we do not show this in the notation. Next, we define the functions

$$\begin{aligned} \mathcal{A}_{\tau,z}(x) &= b_{\tau,z}(x)-\lambda^{1/6}\int_0^\infty \mathrm{Ai}(x+y+\sigma)\widetilde{b}_{\tau,z}(y)\,\mathrm{d}y\\ \widetilde{\mathcal{A}}_{\tau,z}(x) &= \widetilde{b}_{\tau,z}(x)-\lambda^{-1/6}\int_0^\infty \mathrm{Ai}(x+y+\sigma)b_{\tau,z}(y)\,\mathrm{d}y. \end{aligned} \tag{2.21}$$

Again we have $\widetilde{\mathcal{A}}_{\tau,z}(x)=\mathcal{A}_{\tau,-z}(x)$ in the symmetric case $\lambda=1$, and we suppress the dependence on λ,σ from the notation.

We are now ready to introduce the tacnode kernel $\mathcal{L}_{\mathrm{tac}}(u,v)=\mathcal{L}_{\mathrm{tac}}(u,v;\sigma,\tau)$ of Ferrari-Vető [16]‡, restricted to the single-time case $\tau_1=\tau_2=\tau$. The kernel can be represented in several equivalent ways. We find it convenient to use the following representation.

†The notations $b^{\lambda}_{\tau,\sigma+\xi}(x+\widetilde{\sigma})$ and $b^{\lambda^{-1}}_{\lambda^{1/3}\tau,\lambda^{2/3}\sigma-\lambda^{1/6}\xi}(x+\widetilde{\sigma})$ in [16] correspond to our notations $\lambda^{1/6}\widetilde{b}_{-\tau,\xi}(x)$ and $\lambda^{-1/6}b_{-\tau,\xi}(x)$ respectively.

‡This kernel is called $\mathcal{L}^{\lambda,\sigma}_{\mathrm{tac}}(\tau_1,\xi_1,\tau_2,\xi_2)$ in [16], with $\xi_1=u$, $\xi_2=v$ and $\tau_1=\tau_2=\tau$. Recall that we use σ in a different meaning.

Proposition 2.4. *Fix $\lambda > 0$. The tacnode kernel $\mathcal{L}_{\rm tac}(u,v)$ of Ferrari-Vető [16] in the single-time case $\tau_1 = \tau_2 = \tau$ can be written in the form*

$$\begin{aligned}\mathcal{L}_{\rm tac}(u,v;\sigma,\tau) &= \lambda^{1/6} e^{\sqrt{\lambda}\tau(u-v)} K_{{\rm Ai},\lambda^{2/3}(\Sigma+\tau^2)}(-\lambda^{1/6}u, -\lambda^{1/6}v) \\ &\quad + (1+\lambda^{-1/2})^{1/3} \int_0^\infty\int_0^\infty (\mathbf{1} - \mathbf{K}_{{\rm Ai},\sigma})^{-1}(x,y) \mathcal{A}_{\tau,u}(x)\mathcal{A}_{-\tau,v}(y)\,{\rm d}x\,{\rm d}y, \end{aligned} \tag{2.22}$$

with the notations (2.14) *and* (2.17)–(2.21).

Proposition 2.4 is proved in Section 4.1. It was obtained in the symmetric case $\lambda = 1$ by Adler, Johansson and van Moerbeke [3, Theorem 1.2(i)].

In the above proposition we use the notation (2.17). Hence the double integral can be rewritten as

$$\begin{aligned}&\int_0^\infty\int_0^\infty (\mathbf{1} - \mathbf{K}_{{\rm Ai},\sigma})^{-1}(x,y)\, \mathcal{A}_{\tau,u}(x)\mathcal{A}_{-\tau,v}(y)\,{\rm d}x\,{\rm d}y \\ &\qquad = \int_0^\infty\int_0^\infty R_\sigma(x,y)\mathcal{A}_{\tau,u}(x)\mathcal{A}_{-\tau,v}(y)\,{\rm d}x\,{\rm d}y + \int_0^\infty \mathcal{A}_{\tau,u}(x)\mathcal{A}_{-\tau,v}(x)\,{\rm d}x.\end{aligned}$$

2.4 Derivative of the Ferrari-Vető tacnode kernel

Inspired by Theorem 2.3, we consider the derivative of the tacnode kernel $\mathcal{L}_{\rm tac}(u,v)$ with respect to the parameter σ.

Theorem 2.5. *Fix $\lambda > 0$. The tacnode kernel $\mathcal{L}_{\rm tac}(u,v)$ in* (2.22) *satisfies*

$$\frac{\partial}{\partial\sigma}\mathcal{L}_{\rm tac}(u,v;\sigma,\tau) = -C^{-2}\left(\lambda^{1/3}\widehat{p}_1(u;\sigma,\tau)\widehat{p}_1(v;\sigma,-\tau) + \lambda^{-1/2}\widehat{p}_2(u;\sigma,\tau)\widehat{p}_2(v;\sigma,-\tau)\right) \tag{2.23}$$

and consequently

$$\mathcal{L}_{\rm tac}(u,v;\sigma,\tau) = C^{-2}\int_\sigma^\infty \left(\lambda^{1/3}\widehat{p}_1(u;s,\tau)\widehat{p}_1(v;s,-\tau) + \lambda^{-1/2}\widehat{p}_2(u;s,\tau)\widehat{p}_2(v;s,-\tau)\right){\rm d}s. \tag{2.24}$$

Here we denote $C = (1+\lambda^{-1/2})^{1/3}$ and

$$\begin{aligned}\widehat{p}_1(z;\sigma,\tau) &= \int_0^\infty (\mathbf{1} - \mathbf{K}_{{\rm Ai},\sigma})^{-1}(x,0)\, \widetilde{\mathcal{A}}_{\tau,z}(x)\,{\rm d}x \\ \widehat{p}_2(z;\sigma,\tau) &= \int_0^\infty (\mathbf{1} - \mathbf{K}_{{\rm Ai},\sigma})^{-1}(x,0)\, \mathcal{A}_{\tau,z}(x)\,{\rm d}x,\end{aligned} \tag{2.25}$$

with the notation (2.17).

Theorem 2.5 is proved in Section 4.3. In the proof we will use certain functions (4.8) that were introduced by Tracy-Widom [26]. We will also obtain some alternative representations for (2.25), see Lemma 4.3. Note that the right hand side of (2.23) is again a rank-2 kernel. This is a result of independent interest.

By combining Theorems 2.3 and 2.5, we obtain

Theorem 2.6. *(Connection between tacnode kernels.) Fix $\lambda > 0$. The kernel $\mathcal{L}_{\rm tac}(u,v;\sigma,\tau)$ in* (2.22) *equals the RH kernel $K_{\rm tac}(u,v;r_1,r_2,s_1,s_2,\tau)$ in* (2.8) *with the parameters*

$$r_1 = \lambda^{1/4}, \qquad r_2 = 1, \qquad s_1 = \frac{1}{2}\lambda^{3/4}(\Sigma+\tau^2), \qquad s_2 = \frac{1}{2}(\Sigma+\tau^2), \tag{2.26}$$

where we recall (2.18).

Theorem 2.6 is proved in Section 4.4.

2.5 Airy resolvent formulas for the 4×4 Riemann-Hilbert matrix

In the proof of Theorem 2.6 we will need Airy resolvent formulas for the entries in the first two columns of the RH matrix $\widehat{M}(z) = \widehat{M}(z; r_1, r_2, s_1, s_2, \tau)$. The existence of such formulas could be anticipated by comparing Theorems 2.3 and 2.5.

The formulas below will be stated for general values of r_1, r_2, s_1, s_2, τ. The reader who is only interested in the symmetric case $r_1 = r_2$ and $s_1 = s_2$ can skip the next paragraph and move directly to (2.29).

For general $r_1, r_2 > 0$ and $s_1, s_2, \tau \in \mathbb{R}$, we define the constants

$$\begin{aligned} C &= (r_1^{-2} + r_2^{-2})^{1/3}, \\ D &= \sqrt{\frac{r_1}{r_2}} \exp\left(\frac{r_1^4 - r_2^4}{3} \tau^3 + 2(r_2 s_2 - r_1 s_1)\tau \right), \\ \sigma &= C^{-1} \left(2 \left(\frac{s_1}{r_1} + \frac{s_2}{r_2} \right) - (r_1^2 + r_2^2)\tau^2 \right), \end{aligned} \tag{2.27}$$

and the functions

$$\begin{aligned} b_z(x) &= \sqrt{2\pi} r_2^{1/6} \exp\left(-r_2^2 \tau \,(z + Cx)\right) \operatorname{Ai}\left(r_2^{2/3} \left(z + Cx + 2\frac{s_2}{r_2} \right) \right), \\ \widetilde{b}_z(x) &= \sqrt{2\pi} r_1^{1/6} \exp\left(r_1^2 \tau \,(z - Cx)\right) \operatorname{Ai}\left(r_1^{2/3} \left(-z + Cx + 2\frac{s_1}{r_1} \right) \right). \end{aligned} \tag{2.28}$$

The above definitions of C, σ are consistent with our earlier formulas (2.20) and (2.18) under the identification (2.26). Similarly, the formulas for $b_z(x), \widetilde{b}_z(x)$ reduce to the ones in (2.19) up to certain multiplicative constants (independent of z, x).

In the symmetric case where $r_1 = r_2 = 1$, $s_1 = s_2 =: s$, the above definitions simplify to

$$\begin{aligned} C &= 2^{1/3}, \\ D &= 1, \\ \sigma &= 2^{5/3} s - 2^{2/3} \tau^2, \\ b_z(x) &= \sqrt{2\pi} \exp\left(-\tau \left(z + 2^{1/3} x \right) \right) \operatorname{Ai}\left(z + 2^{1/3} x + 2s \right), \\ \widetilde{b}_z(x) &= b_{-z}(x). \end{aligned} \tag{2.29}$$

The fact of the matter is

Theorem 2.7. *(Airy resolvent formulas for the 4×4 RH matrix.) Denote by $\widehat{M}_{j,k}(z)$ the (j,k) entry of the RH matrix $\widehat{M}(z) = \widehat{M}(z; r_1, r_2, s_1, s_2, \tau)$. Then the entries in the top left 2×2 block of $\widehat{M}(z)$ can be expressed by the formulas*

$$\begin{aligned} \widehat{M}_{1,1}(z) &= \int_0^\infty (\mathbf{1} - \mathbf{K}_{\mathrm{Ai},\sigma})^{-1}(x,0)\, \widetilde{b}_z(x)\,\mathrm{d}x, \\ \widehat{M}_{2,1}(z) &= -D \int_0^\infty \int_0^\infty (\mathbf{1} - \mathbf{K}_{\mathrm{Ai},\sigma})^{-1}(x,0) \operatorname{Ai}(x + y + \sigma) \widetilde{b}_z(y)\,\mathrm{d}x\,\mathrm{d}y, \\ \widehat{M}_{1,2}(z) &= -D^{-1} \int_0^\infty \int_0^\infty (\mathbf{1} - \mathbf{K}_{\mathrm{Ai},\sigma})^{-1}(x,0) \operatorname{Ai}(x + y + \sigma) b_z(y)\,\mathrm{d}x\,\mathrm{d}y, \\ \widehat{M}_{2,2}(z) &= \int_0^\infty (\mathbf{1} - \mathbf{K}_{\mathrm{Ai},\sigma})^{-1}(x,0)\, b_z(x)\,\mathrm{d}x, \end{aligned} \tag{2.30}$$

where we use the notations (2.17) *and* (2.27)–(2.28) *(or* (2.29) *in the symmetric case). The entries in the bottom left 2×2 block of $\widehat{M}(z)$ can be obtained by combining the above expressions with equations* (2.36)–(2.37) *below.*

Theorem 2.7 is proved in Section 5. The proof makes heavy use of the Tracy-Widom functions defined in Section 4.2.

Corollary 2.8. *The first two entries of the vector* $\mathbf{p}(z)$ *in* (2.9) *are given by*

$$\begin{aligned} p_1(z) &= \int_0^\infty (\mathbf{1}-\mathbf{K}_{\mathrm{Ai},\sigma})^{-1}(x,0)\widetilde{\mathcal{A}}_z(x)\,\mathrm{d}x, \\ p_2(z) &= \int_0^\infty (\mathbf{1}-\mathbf{K}_{\mathrm{Ai},\sigma})^{-1}(x,0)\mathcal{A}_z(x)\,\mathrm{d}x, \end{aligned} \tag{2.31}$$

where

$$\begin{aligned} \mathcal{A}_z(x) &= b_z(x) - D\int_0^\infty \mathrm{Ai}(x+y+\sigma)\widetilde{b}_z(y)\,\mathrm{d}y, \\ \widetilde{\mathcal{A}}_z(x) &= \widetilde{b}_z(x) - D^{-1}\int_0^\infty \mathrm{Ai}(x+y+\sigma)b_z(y)\,\mathrm{d}y. \end{aligned} \tag{2.32}$$

2.6 Differential equations for the columns of $M(z)$

In the proof of Theorem 2.7, we will use the differential equations for the columns of the RH matrix $\widehat{M}(z)$ (or $M(z)$). Interestingly, the coefficients in these differential equations contain the Hastings-McLeod solution $q(x)$ to Painlevé II. We also need the associated *Hamiltonian*

$$u(x) := (q'(x))^2 - xq^2(x) - q^4(x). \tag{2.33}$$

Proposition 2.9. *(System of differential equations.)*

(a) Let the vector $\mathbf{m}(z) = \mathbf{m}(z; r_1, r_2, s_1, s_2, \tau)$ *be one of the columns of* $\widehat{M}(z)$*, or a fixed linear combination of them, and denote its entries by* $m_j(z)$*,* $j = 1, \ldots, 4$*. Then with the prime denoting the derivative with respect to* z*, we have*

$$\begin{aligned} r_1^{-2}m_1'' &= 2\tau m_1' + C^2D^{-1}q(\sigma)m_2' \\ &\qquad + \left[Cq^2(\sigma) - z + 2s_1/r_1 - r_1^2\tau^2\right]m_1 - \left[CD^{-1}q'(\sigma)\right]m_2, && (2.34) \\ r_2^{-2}m_2'' &= -C^2Dq(\sigma)m_1' - 2\tau m_2' \\ &\qquad + \left[Cq^2(\sigma) + z + 2s_2/r_2 - r_2^2\tau^2\right]m_2 - \left[CDq'(\sigma)\right]m_1, && (2.35) \\ r_1 i m_3 &= m_1' - (C^{-1}u(\sigma) - s_1^2 + r_1^2\tau)m_1 - C^{-1}D^{-1}q(\sigma)m_2, && (2.36) \\ r_2 i m_4 &= m_2' + C^{-1}Dq(\sigma)m_1 + (C^{-1}u(\sigma) - s_2^2 + r_2^2\tau)m_2, && (2.37) \end{aligned}$$

where the constants C, D, σ *are defined in* (2.27) *and we denote by* q, u *the Hastings-McLeod function and the associated Hamiltonian.*

(b) Conversely, any vector $\mathbf{m}(z)$ *that solves* (2.34)–(2.37) *is a fixed (independent of* z*) linear combination of the columns of* $\widehat{M}(z)$*.*

Proposition 2.9 is proved in Section 6.4 with the help of Lax pair calculations. There are similar differential equations with respect to the parameters s_1, s_2 or τ but they will not be needed.

2.7 Outline of the paper

The remainder of this paper is organized as follows. In Section 3 we establish Theorem 2.3 about the derivative of the RH tacnode kernel. In Section 4 we prove Proposition 2.4 and Theorems 2.5 and 2.6 about the Ferrari-Vető tacnode kernel. In Section 5 we prove Theorem 2.7 about the Airy resolvent formulas for the entries of the RH matrix $\widehat{M}(z)$. Finally, in Section 6 we use Lax pair calculations to prove Proposition 2.9.

3 Proof of Theorem 2.3

Throughout the proof we use the parametrization $s_j = \sigma_j s$ with σ_j fixed, $j = 1, 2$. We also assume $r_1, r_2 > 0$ to be fixed. The RH matrix $\widehat{M}(z) = \widehat{M}(z; s, \tau)$ satisfies the differential equation

$$\frac{\partial}{\partial s}\widehat{M}(z) = V(z)\widehat{M}(z), \tag{3.1}$$

for a certain coefficient matrix $V(z) = V(z; s, \tau)$. This is described in more detail in Section 6.3. At this moment, we only need to know that

$$V(z) = -2iz \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 \end{pmatrix} + V(0), \tag{3.2}$$

where the matrix $V(0)$ is independent of z.

Lemma 3.1. *We have*

$$\begin{aligned} \widehat{M}^{-1}(z; s, \tau) &= K^{-1}\widehat{M}^T(z; s, -\tau)K, & (3.3) \\ V^T(z; s, \tau) &= -KV(z; s, -\tau)K^{-1}, & (3.4) \end{aligned}$$

where the superscript ^{T} *denotes the transpose and*

$$K = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} \tag{3.5}$$

with I_2 *denoting the identity matrix of size* 2×2.

Proof. The first equation follows from Lemma 6.3 below. The second equation then follows by combining (3.3) and (3.1). □

We are now ready to prove Theorem 2.3. We write (2.8) in the form

$$\begin{aligned} K_{\mathrm{tac}}(u, v; s, \tau) &= \frac{1}{2\pi i(u-v)} \begin{pmatrix} 1 & 1 & 0 & 0 \end{pmatrix} \widehat{M}^T(v; s, -\tau) K \widehat{M}(u; s, \tau) \begin{pmatrix} 1 & 1 & 0 & 0 \end{pmatrix}^T \\ &= \frac{1}{2\pi i(u-v)} \mathbf{p}^T(v; s, -\tau) K \mathbf{p}(u; s, \tau), \end{aligned} \tag{3.6}$$

where we used (3.3) and the definition of $\mathbf{p}(z)$ (2.9). From (3.1) we then obtain

$$\begin{aligned} \frac{\partial}{\partial s} K_{\mathrm{tac}}(u, v; s, \tau) &= \frac{1}{2\pi i(u-v)} \mathbf{p}^T(v; s, -\tau) \left[V^T(v; s, -\tau)K + KV(u; s, \tau) \right] \mathbf{p}(u; s, \tau) \\ &= \frac{1}{2\pi i(u-v)} \mathbf{p}^T(v; s, -\tau) K \left[-V(v; s, \tau) + V(u; s, \tau) \right] \mathbf{p}(u; s, \tau) \\ &= -\frac{1}{\pi} \mathbf{p}^T(v; s, -\tau) K \begin{pmatrix} 0 & 0 \\ \Sigma & 0 \end{pmatrix} \mathbf{p}(u; s, \tau) \\ &= -\frac{1}{\pi} \mathbf{p}^T(v; s, -\tau) \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} \mathbf{p}(u; s, \tau), \end{aligned}$$

with $\Sigma := \mathrm{diag}(\sigma_1, \sigma_2)$, where the second equality follows from (3.4), and the third equality uses (3.2). This proves (2.11). By integrating this equality, we obtain (2.12), due to the fact that the entries of $\mathbf{p}(z; s, \tau)$ go to zero for $s \to +\infty$ if $\sigma_1, \sigma_2 > 0$. The latter fact follows from [13, Sec. 3] if $\tau = 0$ and is established similarly for general $\tau \in \mathbb{R}$. This ends the proof of Theorem 2.3. □

4 Proofs of Proposition 2.4 and Theorems 2.5, 2.6

4.1 Proof of Proposition 2.4

Denote by $\mathbf{A}_\sigma$ the operator on $L^2([0,\infty))$ that acts on the function $f \in L^2([0,\infty))$ by the rule

$$[\mathbf{A}_\sigma f](x) = \int_0^\infty \mathrm{Ai}(x+y+\sigma) f(y)\,\mathrm{d}y. \tag{4.1}$$

Observe that

$$\mathbf{A}_\sigma^2 = \mathbf{K}_{\mathrm{Ai},\sigma}, \tag{4.2}$$

on account of (2.14).

Now the kernel $\mathcal{L}_{\mathrm{tac}}(u,v)$ in [16, Eq. (5)] in the single time case $\tau_1=\tau_2=\tau$ has the form[§]

$$\begin{aligned}\mathcal{L}_{\mathrm{tac}}(u,v;\sigma,\tau) \;=\;& e^{\tau(v-u)}K_{\mathrm{Ai},\Sigma+\tau^2}(u,v)\\ &+C\int_0^\infty\!\!\int_0^\infty (\mathbf{1}-\mathbf{K}_{\mathrm{Ai},\sigma})^{-1}(x,y)\,[\mathbf{A}_\sigma b_{\tau,u}](x)\,[\mathbf{A}_\sigma b_{-\tau,v}](y)\,\mathrm{d}x\,\mathrm{d}y\\ &-\lambda^{1/6}C\int_0^\infty\!\!\int_0^\infty (\mathbf{1}-\mathbf{K}_{\mathrm{Ai},\sigma})^{-1}(x,y)\,[\mathbf{A}_\sigma b_{\tau,u}](x)\widetilde{b}_{-\tau,v}(y)\,\mathrm{d}x\,\mathrm{d}y\\ &+\lambda^{1/6}e^{\sqrt{\lambda}\tau(u-v)}K_{\mathrm{Ai},\lambda^{2/3}(\Sigma+\tau^2)}(-\lambda^{1/6}u,-\lambda^{1/6}v)\\ &+\lambda^{1/3}C\int_0^\infty\!\!\int_0^\infty (\mathbf{1}-\mathbf{K}_{\mathrm{Ai},\sigma})^{-1}(x,y)\left[\mathbf{A}_\sigma \widetilde{b}_{\tau,u}\right](x)\left[\mathbf{A}_\sigma \widetilde{b}_{-\tau,v}\right](y)\,\mathrm{d}x\,\mathrm{d}y\\ &-\lambda^{1/6}C\int_0^\infty\!\!\int_0^\infty (\mathbf{1}-\mathbf{K}_{\mathrm{Ai},\sigma})^{-1}(x,y)\left[\mathbf{A}_\sigma \widetilde{b}_{\tau,u}\right](x) b_{-\tau,v}(y)\,\mathrm{d}x\,\mathrm{d}y,\end{aligned} \tag{4.3}$$

where again $C=(1+\lambda^{-1/2})^{1/3}$ and (2.17). The first term in the right hand side of (4.3) equals

$$e^{\tau(v-u)}K_{\mathrm{Ai},\Sigma+\tau^2}(u,v) = C\int_0^\infty b_{\tau,u}(x)b_{-\tau,v}(x)\,\mathrm{d}x, \tag{4.4}$$

on account of (2.19). The second term in the right hand side of (4.3) equals

$$\begin{aligned}&C\int_0^\infty\!\!\int_0^\infty (\mathbf{1}-\mathbf{K}_{\mathrm{Ai},\sigma})^{-1}(x,y)\,[\mathbf{A}_\sigma b_{\tau,u}](x)\,[\mathbf{A}_\sigma b_{-\tau,v}](y)\,\mathrm{d}x\,\mathrm{d}y\\ &=C\int_0^\infty\!\!\int_0^\infty (\mathbf{1}-\mathbf{K}_{\mathrm{Ai},\sigma})^{-1}(x,y)\; b_{\tau,u}(x)b_{-\tau,v}(y)\,\mathrm{d}x\,\mathrm{d}y - C\int_0^\infty b_{\tau,u}(x)b_{-\tau,v}(x)\,\mathrm{d}x,\end{aligned} \tag{4.5}$$

where we used that

$$\mathbf{A}_\sigma(\mathbf{1}-\mathbf{K}_{\mathrm{Ai},\sigma})^{-1}\mathbf{A}_\sigma = (\mathbf{1}-\mathbf{K}_{\mathrm{Ai},\sigma})^{-1}\mathbf{K}_{\mathrm{Ai},\sigma} = (\mathbf{1}-\mathbf{K}_{\mathrm{Ai},\sigma})^{-1}-\mathbf{1},$$

on account of (4.2). Finally, the third term in the right hand side of (4.3) can be written as

$$\begin{aligned}&-\lambda^{1/6}C\int_0^\infty\!\!\int_0^\infty (\mathbf{1}-\mathbf{K}_{\mathrm{Ai},\sigma})^{-1}(x,y)\,[\mathbf{A}_\sigma b_{\tau,u}](x)\widetilde{b}_{-\tau,v}(y)\,\mathrm{d}x\,\mathrm{d}y\\ &\qquad= -\lambda^{1/6}C\int_0^\infty\!\!\int_0^\infty (\mathbf{1}-\mathbf{K}_{\mathrm{Ai},\sigma})^{-1}(x,y)\; b_{\tau,u}(x)\left[\mathbf{A}_\sigma \widetilde{b}_{-\tau,v}\right](y)\,\mathrm{d}x\,\mathrm{d}y,\end{aligned} \tag{4.6}$$

[§]The notations σ, $\widetilde{\sigma}$, $b^{\lambda}_{\tau,\sigma+\xi}(x+\widetilde{\sigma})$, $B^{\lambda}_{\tau,\sigma+\xi}(x+\widetilde{\sigma})$, $b^{\lambda^{-1}}_{\lambda^{1/3}\tau,\lambda^{2/3}\sigma-\lambda^{1/6}\xi}(x+\widetilde{\sigma})$, $B^{\lambda^{-1}}_{\lambda^{1/3}\tau,\lambda^{2/3}\sigma-\lambda^{1/6}\xi}(x+\widetilde{\sigma})$ in [16] correspond to our notations Σ, σ, $\lambda^{1/6}\widetilde{b}_{-\tau,\xi}(x)$, $[\mathbf{A}_\sigma b_{-\tau,\xi}](x)$, $\lambda^{-1/6}b_{-\tau,\xi}(x)$, $\left[\mathbf{A}_\sigma \widetilde{b}_{-\tau,\xi}\right](x)$, respectively.

since the operators $(\mathbf{1}-\mathbf{K}_{\mathrm{Ai},\sigma})^{-1}$ and $\mathbf{A}_\sigma$ commute. Inserting (4.4)–(4.6) for the first three terms in the right hand side of (4.3), we obtain

$$\begin{aligned}
&\mathcal{L}_{\mathrm{tac}}(u,v;\sigma,\tau)\\
=\;&\lambda^{1/6}e^{\sqrt{\lambda}\tau(u-v)}K_{\mathrm{Ai},\lambda^{2/3}(\Sigma+\tau^2)}(-\lambda^{1/6}u,-\lambda^{1/6}v)\\
&+C\int_0^\infty\int_0^\infty(\mathbf{1}-\mathbf{K}_{\mathrm{Ai},\sigma})^{-1}(x,y)\;b_{\tau,u}(x)b_{-\tau,v}(y)\,\mathrm{d}x\,\mathrm{d}y\\
&-\lambda^{1/6}C\int_0^\infty\int_0^\infty(\mathbf{1}-\mathbf{K}_{\mathrm{Ai},\sigma})^{-1}(x,y)\;b_{\tau,u}(x)\left[\mathbf{A}_\sigma\widetilde{b}_{-\tau,v}\right](y)\,\mathrm{d}x\,\mathrm{d}y\\
&+\lambda^{1/3}C\int_0^\infty\int_0^\infty(\mathbf{1}-\mathbf{K}_{\mathrm{Ai},\sigma})^{-1}(x,y)\left[\mathbf{A}_\sigma\widetilde{b}_{\tau,u}\right](x)\left[\mathbf{A}_\sigma\widetilde{b}_{-\tau,v}\right](y)\,\mathrm{d}x\,\mathrm{d}y\\
&-\lambda^{1/6}C\int_0^\infty\int_0^\infty(\mathbf{1}-\mathbf{K}_{\mathrm{Ai},\sigma})^{-1}(x,y)\left[\mathbf{A}_\sigma\widetilde{b}_{\tau,u}\right](x)b_{-\tau,v}(y)\,\mathrm{d}x\,\mathrm{d}y, \qquad (4.7)
\end{aligned}$$

which is equivalent to the desired result (2.22). □

4.2 Tracy-Widom functions and their properties

In some of the remaining proofs we need the functions

$$\begin{aligned}
Q(x)&=\int_0^\infty(\mathbf{1}-\mathbf{K}_{\mathrm{Ai},\sigma})^{-1}(x,y)\mathrm{Ai}(y+\sigma)\,\mathrm{d}y\\
P(x)&=\int_0^\infty(\mathbf{1}-\mathbf{K}_{\mathrm{Ai},\sigma})^{-1}(x,y)\mathrm{Ai}'(y+\sigma)\,\mathrm{d}y,
\end{aligned}\qquad (4.8)$$

with the usual notation (2.17). Note that Q,P depend on σ but we omit this dependence from the notation. The functions Q,P originate from the seminal paper of Tracy-Widom [26]. Below we list some of their properties. These properties can all be found in [26], see also [5, Sec. 3.8] for a text book treatment. One should take into account that our function $R(x,y)=R_\sigma(x,y)$ equals the one in [26], [5, Sec. 3.8] at the shifted arguments $x+\sigma$, $y+\sigma$, and similarly for $Q(x)$ and $P(x)$.

Lemma 4.1. *The functions Q,P and $R=R_\sigma$ satisfy the differential equations*

$$\begin{aligned}
\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)R(x,y)&=R(x,0)R(0,y)-Q(x)Q(y), &(4.9)\\
\frac{\partial}{\partial\sigma}R(x,y)&=-Q(x)Q(y), &(4.10)\\
Q'(x)&=P(x)+qR(x,0)-uQ(x), &(4.11)\\
P'(x)&=(x+\sigma-2v)Q(x)+pR(x,0)+uP(x), &(4.12)
\end{aligned}$$

where

$$\begin{aligned}
q&=Q(0) &(4.13)\\
p&=P(0) &(4.14)\\
u&=\int_0^\infty Q(x)\mathrm{Ai}(x+\sigma)\,\mathrm{d}x &(4.15)\\
v&=\int_0^\infty Q(x)\mathrm{Ai}'(x+\sigma)\,\mathrm{d}x=\int_0^\infty P(x)\mathrm{Ai}(x+\sigma)\,\mathrm{d}x. &(4.16)
\end{aligned}$$

Note that q, p, u, v are all functions of σ, although we do not show this in the notation. They satisfy the following differential equations with respect to σ [26], [5, Sec. 3.8]

$$q' = p - qu \tag{4.17}$$
$$p' = \sigma q + pu - 2qv \tag{4.18}$$
$$u' = -q^2 \tag{4.19}$$
$$v' = -pq. \tag{4.20}$$

It is known that $q = q(\sigma)$ is the Hastings-McLeod solution to the Painlevé II equation (1.1)–(1.2). Moreover, $u = u(\sigma)$ is the Hamiltonian (2.33), and

$$2v = u^2 - q^2. \tag{4.21}$$

Finally, we establish the following lemma.

Lemma 4.2. *For any function b on $[0, \infty)$ we have*

$$\int_0^\infty Q(x)b(x)\,\mathrm{d}x = \int_0^\infty\int_0^\infty (\mathbf{1} - \mathbf{K}_{\mathrm{Ai},\sigma})^{-1}(x, 0)\,\mathrm{Ai}(x + y + \sigma)b(y)\,\mathrm{d}x\,\mathrm{d}y \tag{4.22}$$

and

$$\int_0^\infty\int_0^\infty Q(x)\mathrm{Ai}(x + y + \sigma)b(y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^\infty R(x, 0)b(x)\,\mathrm{d}x, \tag{4.23}$$

with the usual notations $R = R_\sigma$ and (2.17).

Proof. Recall the operator $\mathbf{A}_\sigma$ defined in (4.1)–(4.2) and let δ_0 denote the Dirac delta function at 0. We have

$$\begin{aligned}\int_0^\infty Q(x)b(x)\,\mathrm{d}x &= \int_0^\infty\int_0^\infty (\mathbf{1} - \mathbf{K}_{\mathrm{Ai},\sigma})^{-1}(x, y)\,\left[\mathbf{A}_\sigma\delta_0\right](y)\,b(x)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_0^\infty\int_0^\infty (\mathbf{1} - \mathbf{K}_{\mathrm{Ai},\sigma})^{-1}(x, y)\,\delta_0(y)\,\left[\mathbf{A}_\sigma b\right](x)\,\mathrm{d}x\,\mathrm{d}y,\end{aligned}$$

where we used that the operators $\mathbf{A}_\sigma$ and $\mathbf{K}_{\mathrm{Ai},\sigma}$ commute. This proves (4.22). Next,

$$\begin{aligned}&\int_0^\infty\int_0^\infty Q(x)\mathrm{Ai}(x + y + \sigma)b(y)\,\mathrm{d}x\,\mathrm{d}y \\ =\ &\int_0^\infty\int_0^\infty (\mathbf{1} - \mathbf{K}_{\mathrm{Ai},\sigma})^{-1}(x, y)\,\left[\mathbf{A}_\sigma\delta_0\right](y)\,\left[\mathbf{A}_\sigma b\right](x)\,\mathrm{d}x\,\mathrm{d}y \\ =\ &\int_0^\infty\int_0^\infty R(x, y)\delta_0(y)b(x)\,\mathrm{d}x\,\mathrm{d}y,\end{aligned}$$

where we used that

$$\mathbf{A}_\sigma(\mathbf{1} - \mathbf{K}_{\mathrm{Ai},\sigma})^{-1}\mathbf{A}_\sigma = (\mathbf{1} - \mathbf{K}_{\mathrm{Ai},\sigma})^{-1}\mathbf{K}_{\mathrm{Ai},\sigma} = \mathbf{R}_{\mathrm{Ai},\sigma}.$$

This proves (4.23). □

4.3 Proof of Theorem 2.5

Lemma 4.3. *The formulas* (2.25) *can be written in the equivalent ways*

$$\begin{aligned}
\widehat{p}_1(z;\sigma,\tau) &= \widetilde{\mathcal{A}}_{\tau,z}(0)+\int_0^\infty R(x,0)\widetilde{\mathcal{A}}_{\tau,z}(x)\,\mathrm{d}x && (4.24)\\
&= \widetilde{b}_{\tau,z}(0)-\lambda^{-1/6}\int_0^\infty Q(x)\mathcal{A}_{\tau,z}(x)\,\mathrm{d}x && (4.25)\\
\widehat{p}_2(z;\sigma,\tau) &= \mathcal{A}_{\tau,z}(0)+\int_0^\infty R(x,0)\mathcal{A}_{\tau,z}(x)\,\mathrm{d}x && (4.26)\\
&= b_{\tau,z}(0)-\lambda^{1/6}\int_0^\infty Q(x)\widetilde{\mathcal{A}}_{\tau,z}(x)\,\mathrm{d}x, && (4.27)
\end{aligned}$$

with the notations $R=R_\sigma$ *and* (4.8).

Proof. Immediate from Lemma 4.2 and the definitions. □

Lemma 4.4. *We have*

$$\begin{aligned}
(1+\lambda^{-1/2})\frac{\partial}{\partial\sigma}b_{\tau,z}(x) &= \lambda^{-1/2}\frac{\partial}{\partial x}b_{\tau,z}(x), && (4.28)\\
(1+\lambda^{-1/2})\frac{\partial}{\partial\sigma}\widetilde{b}_{\tau,z}(x) &= \frac{\partial}{\partial x}\widetilde{b}_{\tau,z}(x), && (4.29)\\
(1+\lambda^{-1/2})\frac{\partial}{\partial\sigma}\mathcal{A}_{\tau,z}(x) &= \lambda^{-1/2}\frac{\partial}{\partial x}\mathcal{A}_{\tau,z}(x)+\lambda^{1/6}\mathrm{Ai}(x+\sigma)\widetilde{b}_{\tau,z}(0). && (4.30)
\end{aligned}$$

Proof. The first two formulas are obvious from the definitions (2.18)–(2.20). For the last formula, we calculate

$$\begin{aligned}
&\left[(1+\lambda^{-1/2})\frac{\partial}{\partial\sigma}-\lambda^{-1/2}\frac{\partial}{\partial x}\right]\mathcal{A}_{\tau,z}(x)\\
=\;&\left[(1+\lambda^{-1/2})\frac{\partial}{\partial\sigma}-\lambda^{-1/2}\frac{\partial}{\partial x}\right]\left(b_{\tau,z}(x)-\lambda^{1/6}\int_0^\infty \mathrm{Ai}(x+y+\sigma)\widetilde{b}_{\tau,z}(y)\,\mathrm{d}y\right)\\
=\;&-\lambda^{1/6}\left[(1+\lambda^{-1/2})\frac{\partial}{\partial\sigma}-\lambda^{-1/2}\frac{\partial}{\partial x}\right]\left(\int_0^\infty \mathrm{Ai}(x+y+\sigma)\widetilde{b}_{\tau,z}(y)\,\mathrm{d}y\right)\\
=\;&-\lambda^{1/6}\int_0^\infty\left(\mathrm{Ai}'(x+y+\sigma)\widetilde{b}_{\tau,z}(y)+\mathrm{Ai}(x+y+\sigma)\widetilde{b}'_{\tau,z}(y)\right)\mathrm{d}y\\
=\;&\lambda^{1/6}\mathrm{Ai}(x+\sigma)\widetilde{b}_{\tau,z}(0),
\end{aligned}$$

where the second and third equalities use (4.28) and (4.29) respectively, and the last equality follows from integration by parts. This proves the lemma. □

Now we prove Theorem 2.5. With the help of (4.30), the derivative of (2.22) with

respect to σ becomes

$$
\begin{aligned}
& (1+\lambda^{-1/2})^{2/3}\frac{\partial}{\partial\sigma}\mathcal{L}_{\rm tac}(u,v;\sigma,\tau) \\
= \ & (1+\lambda^{-1/2})^{2/3}\lambda^{1/6}e^{\sqrt{\lambda}\tau(u-v)}\frac{\partial}{\partial\sigma}K_{{\rm Ai},\lambda^{2/3}(\Sigma+\tau^2)}(-\lambda^{1/6}u,-\lambda^{1/6}v) \\
& +\lambda^{-1/2}\int_0^\infty\int_0^\infty(\mathbf{1}-\mathbf{K}_{{\rm Ai},\sigma})^{-1}(x,y)\ \left(\mathcal{A}'_{\tau,u}(x)\mathcal{A}_{-\tau,v}(y)+\mathcal{A}_{\tau,u}(x)\mathcal{A}'_{-\tau,v}(y)\right)\,{\rm d}x\,{\rm d}y \\
& +\lambda^{1/6}\widetilde{b}_{\tau,u}(0)\int_0^\infty Q(y)\mathcal{A}_{-\tau,v}(y)\,{\rm d}y \\
& +\lambda^{1/6}\widetilde{b}_{-\tau,v}(0)\int_0^\infty Q(x)\mathcal{A}_{\tau,u}(x)\,{\rm d}x \\
& +(1+\lambda^{-1/2})\int_0^\infty\int_0^\infty\left(\frac{\partial}{\partial\sigma}R(x,y)\right)\mathcal{A}_{\tau,u}(x)\mathcal{A}_{-\tau,v}(y)\,{\rm d}x\,{\rm d}y,
\end{aligned}
\tag{4.31}
$$

where we used the definition of Q in (4.8). The first term in the right hand side of (4.31) can be written as

$$
\begin{aligned}
& (1+\lambda^{-1/2})^{2/3}\lambda^{1/6}e^{\sqrt{\lambda}\tau(u-v)}\frac{\partial}{\partial\sigma}K_{{\rm Ai},\lambda^{2/3}(\Sigma+\tau^2)}(-\lambda^{1/6}u,-\lambda^{1/6}v) \\
= \ & -\lambda^{1/3}e^{\sqrt{\lambda}\tau(u-v)}{\rm Ai}(-\lambda^{1/6}u+\lambda^{2/3}(\Sigma+\tau^2)){\rm Ai}(-\lambda^{1/6}v+\lambda^{2/3}(\Sigma+\tau^2)) \\
= \ & -\lambda^{1/3}\widetilde{b}_{\tau,u}(0)\widetilde{b}_{-\tau,v}(0),
\end{aligned}
\tag{4.32}
$$

where the first equality follows from (2.18) and (2.14) and the last equality uses (2.19). The second term in the right hand side of (4.31) can be written as

$$
\begin{aligned}
& \lambda^{-1/2}\int_0^\infty\int_0^\infty(\mathbf{1}-\mathbf{K}_{{\rm Ai},\sigma})^{-1}(x,y)\ \left(\mathcal{A}'_{\tau,u}(x)\mathcal{A}_{-\tau,v}(y)+\mathcal{A}_{\tau,u}(x)\mathcal{A}'_{-\tau,v}(y)\right)\,{\rm d}x\,{\rm d}y \\
& = -\lambda^{-1/2}\left(\int_0^\infty\int_0^\infty\left[\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)R(x,y)\right]\mathcal{A}_{\tau,u}(x)\mathcal{A}_{-\tau,v}(y)\,{\rm d}x\,{\rm d}y+\mathcal{A}_{\tau,u}(0)\mathcal{A}_{-\tau,v}(0)\right. \\
& \qquad \left.+\mathcal{A}_{\tau,u}(0)\int_0^\infty R(0,y)\mathcal{A}_{-\tau,v}(y)\,{\rm d}y+\mathcal{A}_{-\tau,v}(0)\int_0^\infty R(x,0)\mathcal{A}_{\tau,u}(x)\,{\rm d}x\right),
\end{aligned}
\tag{4.33}
$$

where we used (2.17) and integration by parts. Finally, we observe that

$$
\left[(1+\lambda^{-1/2})\frac{\partial}{\partial\sigma}-\lambda^{-1/2}\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)\right]R(x,y) \ = \ -\lambda^{-1/2}R(x,0)R(0,y)\ -\ Q(x)Q(y),
\tag{4.34}
$$

on account of (4.9)–(4.10). Inserting (4.32)–(4.34) in the right hand side of (4.31), we obtain

$$
\begin{aligned}
& (1+\lambda^{-1/2})^{2/3}\frac{\partial}{\partial\sigma}\mathcal{L}_{\rm tac}(u,v;\sigma,\tau) \\
= \ & -\lambda^{1/3}\left(\widetilde{b}_{\tau,u}(0)-\lambda^{-1/6}\int_0^\infty Q(x)\mathcal{A}_{\tau,u}(x)\,{\rm d}x\right)\left(\widetilde{b}_{-\tau,v}(0)-\lambda^{-1/6}\int_0^\infty Q(x)\mathcal{A}_{-\tau,v}(x)\,{\rm d}x\right) \\
& -\lambda^{-1/2}\left(\mathcal{A}_{\tau,u}(0)+\int_0^\infty R(x,0)\mathcal{A}_{\tau,u}(x)\,{\rm d}x\right)\left(\mathcal{A}_{-\tau,v}(0)+\int_0^\infty R(0,x)\mathcal{A}_{-\tau,v}(x)\,{\rm d}x\right) \\
= \ & -\lambda^{1/3}\widehat{p}_1(u;\sigma,\tau)\widehat{p}_1(v;\sigma,-\tau)-\lambda^{-1/2}\widehat{p}_2(u;\sigma,\tau)\widehat{p}_2(v;\sigma,-\tau),
\end{aligned}
\tag{4.35}
$$

on account of (4.25)–(4.26). This proves (2.23).

Finally we prove (2.24). From [5, Sec. 3.8] we have the estimate

$$|R_\sigma(x,y)| < C_0 e^{-x-y-2\sigma} \tag{4.36}$$

for all $x, y, \sigma > 0$, with $C_0 > 0$ a certain constant. We also have the asymptotics for the Airy function

$$\mathrm{Ai}(x) \sim \exp\left(-\frac{2}{3}x^{3/2}\right)/(2\sqrt{\pi}x^{1/4}), \qquad x \to +\infty. \tag{4.37}$$

Consequently the kernel $\mathcal{L}_{\mathrm{tac}}(u,v;\sigma,\tau)$ in (2.22), (2.17) goes to zero (at a very fast rate) for $\sigma \to +\infty$. Integration of (2.23) then yields (2.24). This proves Theorem 2.5. $\square$

4.4 Proof of Theorem 2.6

Let the parameters r_1, r_2, s_1, s_2 be given by (2.26). As already observed, the expressions for C, σ in (2.27) and (2.20), (2.18) are equal under this identification. Similarly, the expressions for p_1, p_2 in (2.31) and $\widehat{p}_1, \widehat{p}_2$ in (2.25) are related by

$$p_j(z) = \sqrt{2\pi} r_j^{1/6} \exp\left(r_j^4 \tau(\Sigma + \frac{2}{3}\tau^2)\right) \widehat{p}_j(z), \qquad j = 1, 2. \tag{4.38}$$

Note that the exponential factor in (4.38) cancels out in (2.11).

Now observe that the formulas for s_1, s_2 in (2.26) are of the form $s_j = \sigma_j s$ with

$$s := (\Sigma + \tau^2)/2, \qquad \sigma_1 := \lambda^{3/4}, \qquad \sigma_2 := 1. \tag{4.39}$$

By (2.18) we then have

$$\frac{\partial}{\partial s}\mathcal{L}_{\mathrm{tac}}(u,v;\sigma,\tau) = 2\sqrt{\lambda}(1+\lambda^{-1/2})^{2/3}\frac{\partial}{\partial \sigma}\mathcal{L}_{\mathrm{tac}}(u,v;\sigma,\tau), \tag{4.40}$$

where we consider τ to be fixed. Theorem 2.6 follows from Theorems 2.3 and 2.5 and the above observations. $\square$

5 Proof of Theorem 2.7

5.1 Preparations

Lemma 5.1. *The functions $b_z(x)$, $\widetilde{b}_z(x)$ in* (2.28) *satisfy the differential equations*

$$b_z'(x) = C\frac{\partial}{\partial z}b_z(x), \qquad \widetilde{b}_z'(x) = -C\frac{\partial}{\partial z}\widetilde{b}_z(x), \tag{5.1}$$

and

$$r_2^{-2}\frac{\partial^2}{\partial z^2}b_z(x) + 2\tau\frac{\partial}{\partial z}b_z(x) = \left(z + Cx + 2\frac{s_2}{r_2} - r_2^2\tau^2\right)b_z(x), \tag{5.2}$$

$$r_1^{-2}\frac{\partial^2}{\partial z^2}\widetilde{b}_z(x) - 2\tau\frac{\partial}{\partial z}\widetilde{b}_z(x) = \left(-z + Cx + 2\frac{s_1}{r_1} - r_1^2\tau^2\right)\widetilde{b}_z(x). \tag{5.3}$$

Proof. Equation (5.1) is obvious. The other equations follow from a straightforward calculation using the Airy differential equation $\mathrm{Ai}''(x) = x\mathrm{Ai}(x)$. $\square$

Lemma 5.2. *The function* $\mathcal{A}_z(x)$ *in* (2.32) *satisfies the differential equations*

$$\frac{\partial}{\partial z}\mathcal{A}_z(x) = C^{-1}\left(\mathcal{A}_z'(x) - D\mathrm{Ai}(x+\sigma)\widetilde{b}_z(0)\right) \tag{5.4}$$

and

$$\begin{aligned} r_2^{-2}\frac{\partial^2}{\partial z^2}\mathcal{A}_z(x) + 2\tau\frac{\partial}{\partial z}\mathcal{A}_z(x) = &\left(z + Cx + 2\frac{s_2}{r_2} - r_2^2\tau^2\right)\mathcal{A}_z(x) \\ &+ CD\left(\mathrm{Ai}(x+\sigma)\widetilde{b}_z'(0) - \mathrm{Ai}'(x+\sigma)\widetilde{b}_z(0)\right). \end{aligned} \tag{5.5}$$

Proof. Equation (5.4) follows from the definition of $\mathcal{A}_z$, (5.1) and integration by parts. Now we check the formula (5.5). From (2.32) we have

$$\begin{aligned} & r_2^{-2}\frac{\partial^2}{\partial z^2}\mathcal{A}_z(x) \\ = \;& r_2^{-2}\frac{\partial^2}{\partial z^2}b_z(x) - r_2^{-2}D\int_0^\infty \mathrm{Ai}(x+y+\sigma)\frac{\partial^2}{\partial z^2}\widetilde{b}_z(y)\,\mathrm{d}y \\ = \;& r_2^{-2}\frac{\partial^2}{\partial z^2}b_z(x) + r_1^{-2}D\int_0^\infty \mathrm{Ai}(x+y+\sigma)\frac{\partial^2}{\partial z^2}\widetilde{b}_z(y)\,\mathrm{d}y - CD\int_0^\infty \mathrm{Ai}(x+y+\sigma)\widetilde{b}_z''(y)\,\mathrm{d}y \end{aligned}$$

where in the second equality we used $r_1^{-2} + r_2^{-2} = C^3$ and (5.1). Hence

$$\begin{aligned} r_2^{-2}\frac{\partial^2}{\partial z^2}\mathcal{A}_z(x) + 2\tau\frac{\partial}{\partial z}\mathcal{A}_z(x) = \left(r_2^{-2}\frac{\partial^2}{\partial z^2}b_z(x) + 2\tau\frac{\partial}{\partial z}b_z(x)\right) \\ + D\int_0^\infty \mathrm{Ai}(x+y+\sigma)\left(r_1^{-2}\frac{\partial^2}{\partial z^2}\widetilde{b}_z(y) - 2\tau\frac{\partial}{\partial z}\widetilde{b}_z(y)\right)\mathrm{d}y - CD\int_0^\infty \mathrm{Ai}(x+y+\sigma)\widetilde{b}_z''(y)\,\mathrm{d}y. \end{aligned} \tag{5.6}$$

In the first two terms in the right hand side of (5.6) we use the differential equations (5.2)–(5.3), and in the third term we integrate by parts twice and subsequently use the Airy differential equation $\mathrm{Ai}''(x) = x\mathrm{Ai}(x)$. The lemma then follows from a straightforward calculation, taking into account that

$$\left(-z + Cy + 2\frac{s_1}{r_1} - r_1^2\tau^2\right) - C(x+y+\sigma) = -z - Cx - 2\frac{s_2}{r_2} + r_2^2\tau^2 \tag{5.7}$$

thanks to (2.27). □

Lemma 5.3. *The formulas* (2.31) *can be written in the equivalent ways*

$$\begin{aligned} p_1(z) &= \widetilde{\mathcal{A}}_z(0) + \int_0^\infty R(x,0)\widetilde{\mathcal{A}}_z(x)\,\mathrm{d}x && (5.8) \\ &= \widetilde{b}_z(0) - D^{-1}\int_0^\infty Q(x)\mathcal{A}_z(x)\,\mathrm{d}x, && (5.9) \\ p_2(z) &= \mathcal{A}_z(0) + \int_0^\infty R(x,0)\mathcal{A}_z(x)\,\mathrm{d}x && (5.10) \\ &= b_z(0) - D\int_0^\infty Q(x)\widetilde{\mathcal{A}}_z(x)\,\mathrm{d}x. && (5.11) \end{aligned}$$

Proof. Immediate from Lemma 4.2 and the definitions. □

5.2 Differential equation for p_j

In this section we check that the expressions for $p_1(z)$, $p_2(z)$ in (5.9)–(5.10) satisfy the differential equation (2.34) (with $m_j := p_j$). From (5.9) we have

$$p_1'(z) = \frac{\partial}{\partial z}\left[\widetilde{b}_z(0)\right] - D^{-1}\int_0^\infty Q(x)\frac{\partial}{\partial z}\left[\mathcal{A}_z(x)\right]\,\mathrm{d}x. \tag{5.12}$$

We calculate the second term

$$\begin{aligned}
&\int_0^\infty Q(x)\frac{\partial}{\partial z}\left[\mathcal{A}_z(x)\right]\,\mathrm{d}x \\
&= C^{-1}\left(\int_0^\infty Q(x)\mathcal{A}_z'(x)\,\mathrm{d}x - Du\widetilde{b}_z(0)\right) \\
&= -C^{-1}\left(\int_0^\infty Q'(x)\mathcal{A}_z(x)\,\mathrm{d}x + q\mathcal{A}_z(0) + Du\widetilde{b}_z(0)\right) \\
&= -C^{-1}\left(\int_0^\infty \left[P(x) + qR(x,0) - uQ(x)\right]\mathcal{A}_z(x)\,\mathrm{d}x + q\mathcal{A}_z(0) + Du\widetilde{b}_z(0)\right) \\
&= -C^{-1}\left(Dup_1(z) + qp_2(z) + \int_0^\infty P(x)\mathcal{A}_z(x)\,\mathrm{d}x\right)
\end{aligned} \tag{5.13}$$

where the first equality uses (5.4) and (4.15), the second one uses integration by parts and (4.13), the third one uses (4.11), and the fourth equality uses (5.9)–(5.10).

From (5.12)–(5.13) and $r_1^{-2} + r_2^{-2} = C^3$, we get

$$\begin{aligned}
r_1^{-2}p_1'(z) = r_1^{-2}\frac{\partial}{\partial z}[\widetilde{b}_z(0)] + r_2^{-2}D^{-1}\int_0^\infty Q(x)\frac{\partial}{\partial z}\left[\mathcal{A}_z(x)\right]\,\mathrm{d}x \\
+ C^2D^{-1}\left(Dup_1(z) + qp_2(z) + \int_0^\infty P(x)\mathcal{A}_z(x)\,\mathrm{d}x\right).
\end{aligned} \tag{5.14}$$

By differentiating (5.14) with respect to z, we get

$$\begin{aligned}
r_1^{-2}p_1''(z) - C^2D^{-1}qp_2'(z) - 2\tau p_1'(z) = r_1^{-2}\frac{\partial^2}{\partial z^2}\left[\widetilde{b}_z(0)\right] - 2\tau\frac{\partial}{\partial z}\left[\widetilde{b}_z(0)\right] \\
+ D^{-1}\int_0^\infty Q(x)\left(r_2^{-2}\frac{\partial^2}{\partial z^2}\left[\mathcal{A}_z(x)\right] + 2\tau\frac{\partial}{\partial z}\left[\mathcal{A}_z(x)\right]\right)\mathrm{d}x \\
C^2D^{-1}\left(Dup_1'(z) + \int_0^\infty P(x)\frac{\partial}{\partial z}\left[\mathcal{A}_z(x)\right]\,\mathrm{d}x\right),
\end{aligned}$$

where the last term in the left hand side was expanded using (5.12). Equivalently,

$$\begin{aligned}
r_1^{-2}p_1''(z) - C^2D^{-1}qp_2'(z) - 2\tau p_1'(z) = r_1^{-2}\frac{\partial^2}{\partial z^2}\left[\widetilde{b}_z(0)\right] - 2\tau\frac{\partial}{\partial z}\left[\widetilde{b}_z(0)\right] \\
+ CD^{-1}\left(C^{-1}\int_0^\infty Q(x)\left(r_2^{-2}\frac{\partial^2}{\partial z^2}\left[\mathcal{A}_z(x)\right] + 2\tau\frac{\partial}{\partial z}\left[\mathcal{A}_z(x)\right]\right)\mathrm{d}x\right. \\
\left. + CDup_1'(z) + C\int_0^\infty P(x)\frac{\partial}{\partial z}\left[\mathcal{A}_z(x)\right]\,\mathrm{d}x\right).
\end{aligned} \tag{5.15}$$

We will calculate each of the terms in the right hand side of (5.15). We start with

$$\begin{aligned}
C\int_0^\infty P(x)\frac{\partial}{\partial z}\left[\mathcal{A}_z(x)\right]\,\mathrm{d}x &= \int_0^\infty P(x)\mathcal{A}_z'(x)\,\mathrm{d}x - Dv\widetilde{b}_z(0) \\
&= -\left(\int_0^\infty P'(x)\mathcal{A}_z(x)\,\mathrm{d}x + p\mathcal{A}_z(0) + Dv\widetilde{b}_z(0)\right)
\end{aligned}$$

where the first equality follows from (5.4) and (4.16), and the second equality uses integration by parts and (4.14). Consequently,

$$C\int_0^\infty P(x)\frac{\partial}{\partial z}\left[\mathcal{A}_z(x)\right]\,\mathrm{d}x = -\int_0^\infty (x+\sigma)Q(x)\mathcal{A}_z(x)\,\mathrm{d}x \\ - u\int_0^\infty P(x)\mathcal{A}_z(x)\,\mathrm{d}x - pp_2(z) - 2Dvp_1(z) + Dv\widetilde{b}_z(0) \quad (5.16)$$

by virtue of (4.12) and (5.9)–(5.10).

Next we calculate the third term in the right hand side of (5.15),

$$C^{-1}\int_0^\infty Q(x)\left(r_2^{-2}\frac{\partial^2}{\partial z^2}\left[\mathcal{A}_z(x)\right] + 2\tau\frac{\partial}{\partial z}\left[\mathcal{A}_z(x)\right]\right)\,\mathrm{d}x \\ = C^{-1}\int_0^\infty \left(z + Cx + 2\frac{s_2}{r_2} - r_2^2\tau^2\right)Q(x)\mathcal{A}_z(x)\,\mathrm{d}x + D(u\widetilde{b}_z'(0) - v\widetilde{b}_z(0)), \quad (5.17)$$

on account of (5.5) and (4.15)–(4.16). By adding (5.16)–(5.17) and canceling terms we get

$$C\int_0^\infty P(x)\frac{\partial}{\partial z}\left[\mathcal{A}_z(x)\right]\,\mathrm{d}x + C^{-1}\int_0^\infty Q(x)\left(r_2^{-2}\frac{\partial^2}{\partial z^2}\left[\mathcal{A}_z(x)\right] + 2\tau\frac{\partial}{\partial z}\left[\mathcal{A}_z(x)\right]\right)\,\mathrm{d}x \\ = -u\int_0^\infty P(x)\mathcal{A}_z(x)\,\mathrm{d}x + C^{-1}\left(z - 2\frac{s_1}{r_1} + r_1^2\tau^2\right)\int_0^\infty Q(x)\mathcal{A}_z(x)\,\mathrm{d}x \\ + Du\widetilde{b}_z'(0) - 2Dvp_1(z) - pp_2(z), \quad (5.18)$$

where the factor between brackets in front of the second integral in the right hand side was obtained via (5.7). Finally, we calculate the fourth term in the right hand side of (5.15),

$$\begin{aligned} CDup_1'(z) &= CDu\frac{\partial}{\partial z}[\widetilde{b}_z(0)] + Du^2p_1(z) + uqp_2(z) + u\int_0^\infty P(x)\mathcal{A}_z(x)\,\mathrm{d}x \\ &= -Du\widetilde{b}_z'(0) + Du^2p_1(z) + uqp_2(z) + u\int_0^\infty P(x)\mathcal{A}_z(x)\,\mathrm{d}x \qquad (5.19) \end{aligned}$$

where the first equality follows from (5.12)–(5.13) and the second one from (5.1). Inserting (5.18)–(5.19) in the right hand side of (5.15) and canceling terms we get

$$\begin{aligned} & r_1^{-2}p_1''(z) - C^2D^{-1}qp_2'(z) - 2\tau p_1'(z) \\ = & \left(r_1^{-2}\frac{\partial^2}{\partial z^2}\left[\widetilde{b}_z(0)\right] - 2\tau\frac{\partial}{\partial z}\left[\widetilde{b}_z(0)\right]\right) + D^{-1}\left(z - 2\frac{s_1}{r_1} + r_1^2\tau^2\right)\int_0^\infty Q(x)\mathcal{A}_z(x)\,\mathrm{d}x \\ & +C[u^2 - 2v]p_1(z) + CD^{-1}[uq - p]p_2(z) \\ = & \left[-z + 2\frac{s_1}{r_1} - r_1^2\tau^2 + C(u^2 - 2v)\right]p_1(z) + CD^{-1}[uq - p]p_2(z) \\ = & \left[-z + 2\frac{s_1}{r_1} - r_1^2\tau^2 + Cq^2\right]p_1(z) - \left[CD^{-1}q'\right]p_2(z) \end{aligned}$$

where the second equality follows from (5.9) and (5.3), and the last equality uses (4.17) and (4.21). We have established the desired differential equation (2.34).

5.3 Other differential equations

Denote by $\widehat{N}_{j,k}(z)$ the right hand side of the formula for $\widehat{M}_{j,k}(z)$ in (2.30). Let the vectors $\mathbf{p}(z), \mathbf{m}(z)$ have entries

$$p_j(z) = \widehat{N}_{j,1}(z) + \widehat{N}_{j,2}(z), \qquad m_j(z) = \widehat{N}_{j,1}(z) - \widehat{N}_{j,2}(z),$$

for $j = 1, \ldots, 4$. For $\mathbf{p}(z)$ this is compatible with (2.31). We have already shown in Section 5.2 that $\mathbf{p}(z)$ satisfies the differential equation (2.34) (with $m_j := p_j$). We claim that the same statement holds for $\mathbf{m}(z)$. This can be shown by going through the proofs in Sections 5.1–5.2 again and replacing the appropriate plus signs by minus signs and vice versa. We leave this to the reader. Summarizing, both $\mathbf{p}(z)$ and $\mathbf{m}(z)$ satisfy the differential equation (2.34). By symmetry they also satisfy the differential equation (2.35). Finally, (2.36)–(2.37) are valid by construction. Proposition 2.9(b) implies that $\mathbf{p}(z)$ and $\mathbf{m}(z)$ are fixed linear combinations of the columns of $\widehat{M}(z)$. By linearity, the same holds for the $\widehat{N}_{j,k}(z)$.

5.4 Asymptotics

In view of the previous section, Theorem 2.7 will be proved if we can show that the expressions for $\widehat{M}_{j,1}(z)$ and $\widehat{M}_{j,2}(z)$ in (2.30) and (2.36)–(2.37) satisfy the asymptotics for $z \to \infty$ in the RH problem 2.1. It will be enough to prove the asymptotics for the second column $\widehat{M}_{j,2}(z)$. Moreover, it is sufficient to let z go to infinity along the positive real line. Indeed, first observe that the second columns of $M(z)$ and $\widehat{M}(z)$ are equal if $\operatorname{Re} z > 0$. Furthermore, the second column of $M(z)$ is recessive with respect to the other columns as $z \to +\infty$, due to (2.5). So if we can prove that the expressions for $\widehat{M}_{j,2}(z)$ in (2.30) share this same recessive asymptotic behavior for $z \to +\infty$, then we are done.

The proof of the above asymptotics is now an easy consequence of (4.36)–(4.37). This ends the proof of Theorem 2.7.

6 Proof of Proposition 2.9

6.1 Painlevé formulas for the residue matrix M_1

Let the parameters $r_1, r_2 > 0$ and $s_1, s_2, \tau \in \mathbb{R}$ be fixed. In the proof of Proposition 2.9 we will need Painlevé formulas for the entries of the 'residue' matrix $M_1 = M_1(r_1, r_2, s_1, s_2, \tau)$ in (2.5). Write this matrix in entrywise form as

$$M_1 =: \begin{pmatrix} a & b & ic & id \\ -\widetilde{b} & -\widetilde{a} & i\widetilde{d} & i\widetilde{c} \\ ie & i\widetilde{f} & -\alpha & \widetilde{\beta} \\ if & i\widetilde{e} & -\beta & \widetilde{\alpha} \end{pmatrix} \tag{6.1}$$

for certain numbers $a, \widetilde{a}, b, \widetilde{b}, c, \widetilde{c}, \ldots$ that depend on r_1, r_2, s_1, s_2, τ. We will sometimes write $a(r_1, r_2, s_1, s_2, \tau)$, $b(r_1, r_2, s_1, s_2, \tau)$, etc., to denote the dependence on the parameters.

In the symmetric case $r_1 = r_2$, $s_1 = s_2$ we are allowed to drop all the tildes from (6.1), while in the case $\tau = 0$ we can replace all the Greek letters in (6.1) by their Roman counterparts and put $\widetilde{d} = d$ and $\widetilde{f} = f$. These are special instances of the next lemma.

Lemma 6.1. *(Symmetry relations.) Let $r_1, r_2 > 0$ and $s_1, s_2, \tau \in \mathbb{R}$ be fixed. The 16 entries $x = x(r_1, r_2, s_1, s_2, \tau)$ of the matrix* (6.1) *are all real-valued and they satisfy the symmetry relations*

$$x(r_1, r_2, s_1, s_2, \tau) = \widetilde{x}(r_2, r_1, s_2, s_1, \tau), \tag{6.2}$$

for any $x = a, b, c, d, e, f, \alpha, \beta$, and

$$x(r_1, r_2, s_1, s_2, \tau) = \chi(r_1, r_2, s_1, s_2, -\tau), \tag{6.3}$$

for any $x = a, b, \widetilde{a}, \widetilde{b}, c, \widetilde{c}, d, e, \widetilde{e}, f$, where we write $\chi = \alpha, \beta, \widetilde{\alpha}, \widetilde{\beta}, c, \widetilde{c}, \widetilde{d}, e, \widetilde{e}, \widetilde{f}$, respectively.

The proof of Lemma 6.1 follows from Section 6.2 below.

Now we relate the entries of the matrix M_1 to the Hastings-McLeod solution $q(x)$ of Painlevé II and the Hamiltonian $u(x)$ in (2.33). The next theorem was proved for the special case $\tau = 0$ in [13] and in the symmetric setting $r_1 = r_2$, $s_1 = s_2$ in [11, 14]. In the general case we have the extra exponential factor D in (2.27).

Theorem 6.2. *Let the parameters $r_1, r_2 > 0$ and $s_1, s_2, \tau \in \mathbb{R}$ be fixed. The entries in the top right 2×2 block of* (6.1) *are given by*

$$\begin{aligned} d &= (r_2 C D)^{-1} q(\sigma) && (6.4)\\ \widetilde{d} &= (r_1 C)^{-1} D q(\sigma) && (6.5)\\ c &= r_1^{-1}(s_1^2 - C^{-1} u(\sigma)) && (6.6)\\ \widetilde{c} &= r_2^{-1}(s_2^2 - C^{-1} u(\sigma)) && (6.7) \end{aligned}$$

where q is the Hastings-McLeod solution to Painlevé II (1.1)–(1.2)*, u is the Hamiltonian* (2.33)*, and with the constants C, D, σ given by* (2.27)*. Moreover, some of the other entries in* (6.1) *are given by*

$$\begin{aligned} b &= (\widetilde{c} + \tau r_2) d - (r_2^2 C^2 D)^{-1} q'(\sigma) && (6.8)\\ \widetilde{b} &= (c + \tau r_1) \widetilde{d} - (r_1^2 C^2)^{-1} D q'(\sigma) && (6.9)\\ \beta &= (\widetilde{c} - \tau r_2) \widetilde{d} - (r_1 r_2 C^2)^{-1} D q'(\sigma) && (6.10)\\ \widetilde{\beta} &= (c - \tau r_1) d - (r_1 r_2 C^2 D)^{-1} q'(\sigma) && (6.11) \end{aligned}$$

and

$$\begin{aligned} r_1 f &= -\frac{r_2}{r_1^2 + r_2^2} \frac{\partial d}{\partial \tau} + (-r_1 c - r_2 \widetilde{c} + r_1^2 \tau + r_2^2 \tau) b - r_1 d^2 \widetilde{d} + r_2 \widetilde{c}^2 d - 2 s_2 d && (6.12)\\ r_2 \widetilde{f} &= -\frac{r_1}{r_1^2 + r_2^2} \frac{\partial \widetilde{d}}{\partial \tau} + (-r_1 c - r_2 \widetilde{c} + r_1^2 \tau + r_2^2 \tau) \widetilde{b} - r_2 \widetilde{d}^2 d + r_1 c^2 \widetilde{d} - 2 s_1 \widetilde{d}. && (6.13) \end{aligned}$$

Theorem 6.2 is proved in Section 6.5.

6.2 Symmetry relations

For further use, we collect some elementary results concerning symmetry.

Lemma 6.3. *(Symmetry relations.) For any fixed r_1, r_2, s_1, s_2, τ, the RH matrix M satisfies the symmetry relations*

$$\overline{M(\overline{z}; r_1, r_2, s_1, s_2, \tau)} = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} M(z; \overline{r_1}, \overline{r_2}, \overline{s_1}, \overline{s_2}, \overline{\tau}) \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \tag{6.14}$$

where the bar denotes complex conjugation,

$$M^{-T}(z;r_1,r_2,s_1,s_2,\tau)=K^{-1}M(z;r_1,r_2,s_1,s_2,-\tau)K, \tag{6.15}$$

where the superscript $^{-T}$ *denotes the inverse transpose, and finally*

$$M(-z;r_1,r_2,s_1,s_2,\tau)=\begin{pmatrix}J & 0\\ 0 & -J\end{pmatrix}M(z;r_2,r_1,s_2,s_1,\tau)\begin{pmatrix}J & 0\\ 0 & -J\end{pmatrix}, \tag{6.16}$$

where we denote

$$J=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\qquad K=\begin{pmatrix}0 & I_2\\ -I_2 & 0\end{pmatrix}. \tag{6.17}$$

Proof. This follows as in [13, Sec. 5.1]. One easily checks that the left and right hand sides of (6.14) satisfy the same RH problem. Then (6.14) follows from the uniqueness of the solution to this RH problem. The same argument applies to (6.15) and (6.16). □

Corollary 6.4. *For any fixed* r_1,r_2,s_1,s_2,τ*, the residue matrix* M_1 *in* (2.5), (6.1) *satisfies the symmetry relations*

$$\begin{aligned}
\overline{M_1(r_1,r_2,s_1,s_2,\tau)} &= \begin{pmatrix}I_2 & 0\\ 0 & -I_2\end{pmatrix}M_1(\overline{r_1},\overline{r_2},\overline{s_1},\overline{s_2},\overline{\tau})\begin{pmatrix}I_2 & 0\\ 0 & -I_2\end{pmatrix}, && (6.18)\\
M_1^T(r_1,r_2,s_1,s_2,\tau) &= -K^{-1}M_1(r_1,r_2,s_1,s_2,-\tau)K, && (6.19)\\
M_1(r_1,r_2,s_1,s_2,\tau) &= -\begin{pmatrix}J & 0\\ 0 & -J\end{pmatrix}M_1(r_2,r_1,s_2,s_1,\tau)\begin{pmatrix}J & 0\\ 0 & -J\end{pmatrix}, && (6.20)
\end{aligned}$$

with the notations J,K *in* (6.17).

Lemma 6.1 is an immediate consequence of Corollary 6.4.

6.3 Lax system

To the RH matrix $M(z)$ there is associated a Lax system of differential equations

$$\frac{\partial}{\partial z}M=UM,\qquad \frac{\partial}{\partial s}M=VM,\qquad \frac{\partial}{\partial \tau}M=WM, \tag{6.21}$$

for certain coefficient matrices U,V,W. These matrices were obtained in the symmetric case $r_1=r_2$, $s_1=s_2$ in [11, Sec. 5.3], [14] and for $\tau=0$ in [13, Sec. 5.2]. We will consider the general nonsymmetric case. To take derivatives with respect to s_1 or s_2, we again parameterize $s_j=\sigma_j s$ with σ_1,σ_2 fixed and s variable, as in (2.10).

Lemma 6.5. *In the general nonsymmetric setting, with the parametrization* $s_1=\sigma_1 s$,

$s_2 = \sigma_2 s$, *the coefficient matrices* U, V, W *in* (6.21) *take the form*

$$U = \begin{pmatrix} -r_1 c + r_1^2 \tau & r_2 d & r_1 i & 0 \\ -r_1 \widetilde{d} & r_2 \widetilde{c} - r_2^2 \tau & 0 & r_2 i \\ (r_1 c^2 - r_2 d\widetilde{d} - 2s_1 + r_1 z)i & -(r_1 b + r_2 \widetilde{\beta})i & r_1 c + r_1^2 \tau & r_1 d \\ -(r_1 \beta + r_2 \widetilde{b})i & (r_2 \widetilde{c}^2 - r_1 d\widetilde{d} - 2s_2 - r_2 z)i & -r_2 \widetilde{d} & -r_2 \widetilde{c} - r_2^2 \tau \end{pmatrix}, \tag{6.22}$$

$$V = 2 \begin{pmatrix} \sigma_1 c & \sigma_2 d & -\sigma_1 i & 0 \\ \sigma_1 \widetilde{d} & \sigma_2 \widetilde{c} & 0 & \sigma_2 i \\ \sigma_1(-c^2 + \frac{r_2}{r_1} d\widetilde{d} + \frac{\sigma_1}{r_1} s - z)i & (\sigma_1 b - \sigma_2 \widetilde{\beta})i & -\sigma_1 c & -\sigma_1 d \\ (\sigma_1 \beta - \sigma_2 \widetilde{b})i & \sigma_2(\widetilde{c}^2 - \frac{r_1}{r_2} d\widetilde{d} - \frac{\sigma_2}{r_2} s - z)i & -\sigma_2 \widetilde{d} & -\sigma_2 \widetilde{c} \end{pmatrix}, \tag{6.23}$$

$$W = (r_1^2 + r_2^2) \begin{pmatrix} \frac{r_1^2}{r_1^2 + r_2^2} z & -b & 0 & -di \\ -\widetilde{b} & -\frac{r_2^2}{r_1^2 + r_2^2} z & \widetilde{d} i & 0 \\ 0 & -fi & \frac{r_1^2}{r_1^2 + r_2^2} z & -\widetilde{\beta} \\ \widetilde{f} i & 0 & -\beta & -\frac{r_2^2}{r_1^2 + r_2^2} z \end{pmatrix}, \tag{6.24}$$

with the notations in (6.1).

Proof. This is a routine calculation, that follows similarly as in the above cited references [11, 13, 14]. From the asymptotics (2.5) we have for $z \to \infty$ that

$$\frac{\partial M}{\partial z} M^{-1} = \left(I + \frac{M_1}{z} + \cdots \right) \begin{pmatrix} r_1^2 \tau & 0 & i(r_1 - s_1 z^{-1}) & 0 \\ 0 & -r_2^2 \tau & 0 & i(r_2 + s_2 z^{-1}) \\ i(r_1 z - s_1) & 0 & r_1^2 \tau & 0 \\ 0 & -i(r_2 z + s_2) & 0 & -r_2^2 \tau \end{pmatrix} \left(I - \frac{M_1}{z} + \cdots \right) + O(z^{-1}). \tag{6.25}$$

Since the RH matrix $M(z)$ has constant jumps, the left hand side of (6.25) is an entire function of z. Liouville's theorem implies that it is a polynomial in z. Collecting the polynomial terms in the right hand side of (6.25) we obtain

$$U = \begin{pmatrix} r_1^2 \tau & 0 & i r_1 & 0 \\ 0 & -r_2^2 \tau & 0 & i r_2 \\ i(r_1 z - s_1) & 0 & r_1^2 \tau & 0 \\ 0 & -i(r_2 z + s_2) & 0 & -r_2^2 \tau \end{pmatrix} + iM_1 A - iAM_1, \qquad A := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ r_1 & 0 & 0 & 0 \\ 0 & -r_2 & 0 & 0 \end{pmatrix}.$$

With the help of (6.1) and a small calculation, we then get (6.22). To obtain the $(3, 1)$ and $(4, 2)$ entries of (6.22), we also need the relations

$$a + \alpha = -c^2 + \frac{r_2}{r_1} d\widetilde{d} + \frac{s_1}{r_1}, \tag{6.26}$$

$$\widetilde{a} + \widetilde{\alpha} = -\widetilde{c}^2 + \frac{r_1}{r_2} d\widetilde{d} + \frac{s_2}{r_2}, \tag{6.27}$$

which follow from the fact that the $(1, 3)$ and $(2, 4)$ entries in the z^{-1} coefficient in (6.25) are equal to zero. A similar argument yields (6.23)–(6.24). □

6.4 Proof of Proposition 2.9

Let the vector $\mathbf{m}(z)$ be a solution of $\frac{\partial}{\partial z}\mathbf{m} = U\mathbf{m}$ with U in (6.22). By splitting this equation in 2×2 blocks we get

$$\begin{pmatrix} r_1^{-1}m_1'(z) \\ r_2^{-1}m_2'(z) \end{pmatrix} = \begin{pmatrix} -c+r_1\tau & r_1^{-1}r_2 d \\ -r_1 r_2^{-1}\widetilde{d} & \widetilde{c}-r_2\tau \end{pmatrix} \begin{pmatrix} m_1(z) \\ m_2(z) \end{pmatrix} + i \begin{pmatrix} m_3(z) \\ m_4(z) \end{pmatrix}, \tag{6.28}$$

$$\begin{aligned} \begin{pmatrix} r_1^{-1}m_3'(z) \\ r_2^{-1}m_4'(z) \end{pmatrix} = i &\begin{pmatrix} c^2 - r_1^{-1}r_2 d\widetilde{d} - 2r_1^{-1}s_1 + z & -b - r_1^{-1}r_2\widetilde{\beta} \\ -r_1 r_2^{-1}\beta - \widetilde{b} & \widetilde{c}^2 - r_1 r_2^{-1}d\widetilde{d} - 2r_2^{-1}s_2 - z \end{pmatrix} \begin{pmatrix} m_1(z) \\ m_2(z) \end{pmatrix} \\ &+ \begin{pmatrix} c+r_1\tau & d \\ -\widetilde{d} & -\widetilde{c}-r_2\tau \end{pmatrix} \begin{pmatrix} m_3(z) \\ m_4(z) \end{pmatrix}. \end{aligned} \tag{6.29}$$

From (6.28) and (6.4)–(6.7), we easily get (2.36)–(2.37). To prove the two remaining differential equations, we take the derivative of (6.28) and use (6.29) to get

$$\begin{aligned} \begin{pmatrix} r_1^{-2}m_1''(z) \\ r_2^{-2}m_2''(z) \end{pmatrix} &= \begin{pmatrix} -c+r_1\tau & r_1^{-2}r_2^2 d \\ -r_1^2 r_2^{-2}\widetilde{d} & \widetilde{c}-r_2\tau \end{pmatrix} \begin{pmatrix} r_1^{-1}m_1'(z) \\ r_2^{-1}m_2'(z) \end{pmatrix} \\ &- \begin{pmatrix} c^2 - r_1^{-1}r_2 d\widetilde{d} - 2r_1^{-1}s_1 + z & -b - r_1^{-1}r_2\widetilde{\beta} \\ -r_1 r_2^{-1}\beta - \widetilde{b} & \widetilde{c}^2 - r_1 r_2^{-1}d\widetilde{d} - 2r_2^{-1}s_2 - z \end{pmatrix} \begin{pmatrix} m_1(z) \\ m_2(z) \end{pmatrix} \\ &+ \begin{pmatrix} c+r_1\tau & d \\ -\widetilde{d} & -\widetilde{c}-r_2\tau \end{pmatrix} \left[\begin{pmatrix} r_1^{-1}m_1'(z) \\ r_2^{-1}m_2'(z) \end{pmatrix} - \begin{pmatrix} -c+r_1\tau & r_1^{-1}r_2 d \\ -r_1 r_2^{-1}\widetilde{d} & \widetilde{c}-r_2\tau \end{pmatrix} \begin{pmatrix} m_1(z) \\ m_2(z) \end{pmatrix} \right]. \end{aligned}$$

From this equation and (6.4)–(6.11) we obtain the desired differential equations (2.34)–(2.35). This proves Proposition 2.9(a).

To prove Part (b), let $\mathbf{m}(z)$ satisfy the differential equations (2.34)–(2.37). From the proof of Part (a) above, we see that $\frac{\partial}{\partial z}\mathbf{m} = U\mathbf{m}$ with U in (6.22). But then

$$\frac{\partial}{\partial z}[\widehat{M}^{-1}\mathbf{m}] = \widehat{M}^{-1}(U-U)\mathbf{m} = 0,$$

which implies Proposition 2.9(b). □

6.5 Proof of Theorem 6.2

The matrices U, V, W in (6.21) satisfy the compatibility conditions

$$\frac{\partial U}{\partial s} = \frac{\partial V}{\partial z} - UV + VU \tag{6.30}$$

$$\frac{\partial U}{\partial \tau} = \frac{\partial W}{\partial z} - UW + WU. \tag{6.31}$$

These relations are obtained by calculating the mixed derivatives $\frac{\partial^2}{\partial z\partial s}M = \frac{\partial^2}{\partial s\partial z}M$ and $\frac{\partial^2}{\partial z\partial \tau}M = \frac{\partial^2}{\partial \tau\partial z}M$, respectively, in two different ways.

Lemma 6.6. *Consider the matrices U, V, W in* (6.22)*–*(6.24)*. Then with the expressions* (6.4)*–*(6.13) *in Theorem 6.2, the compatibility conditions* (6.30)*–*(6.31) *are satisfied.*

Proof. This is a lengthy but direct calculation. It is best performed with the help of a symbolic computer program such as Maple. First we consider the compatibility condition with respect to s. Writing the matrix equation (6.30) in entrywise form, with the help of Maple, we obtain the system of equations, with the prime denoting the derivative with respect to s,

$$r_1 d' = 2(\sigma_1 r_2 + \sigma_2 r_1)(\widetilde{c}d - b) + 2(r_1^2 + r_2^2)\sigma_1 \tau d \tag{6.32}$$

$$r_2 d' = 2(\sigma_1 r_2 + \sigma_2 r_1)(cd - \widetilde{\beta}) - 2(r_1^2 + r_2^2)\sigma_2 \tau d \tag{6.33}$$

$$r_1 \widetilde{d}' = 2(\sigma_1 r_2 + \sigma_2 r_1)(\widetilde{c}\widetilde{d} - \beta) - 2(r_1^2 + r_2^2)\sigma_1 \tau \widetilde{d} \tag{6.34}$$

$$r_2 \widetilde{d}' = 2(\sigma_1 r_2 + \sigma_2 r_1)(c\widetilde{d} - \widetilde{b}) + 2(r_1^2 + r_2^2)\sigma_2 \tau \widetilde{d} \tag{6.35}$$

$$r_1 c' = 2(\sigma_1 r_2 + \sigma_2 r_1)d\widetilde{d} + 2\sigma_1^2 s \tag{6.36}$$

$$r_2 \widetilde{c}' = 2(\sigma_1 r_2 + \sigma_2 r_1)d\widetilde{d} + 2\sigma_2^2 s \tag{6.37}$$

and

$$\begin{aligned}(r_1 b + r_2\widetilde{\beta})' = {} & (r_1\widetilde{c} + r_2 c)d' - 2(r_1^2 + r_2^2)\tau\sigma_1(\widetilde{c}d - b) + 2(r_1^2 + r_2^2)\tau\sigma_2(cd - \widetilde{\beta}) \\ & - 2\frac{r_1\sigma_2 + r_2\sigma_1}{r_1 r_2}\Big((r_1^2 + r_2^2)d^2\widetilde{d} + (r_1\sigma_2 + r_2\sigma_1)sd\Big) - 4\sigma_1\sigma_2 sd\end{aligned} \tag{6.38}$$

$$\begin{aligned}(r_1 \beta + r_2\widetilde{b})' = {} & (r_1\widetilde{c} + r_2 c)\widetilde{d}' + 2(r_1^2 + r_2^2)\sigma_1\tau(\widetilde{c}\widetilde{d} - \beta) - 2(r_1^2 + r_2^2)\sigma_2\tau(c\widetilde{d} - \widetilde{b}) \\ & - 2\frac{r_1\sigma_2 + r_2\sigma_1}{r_1 r_2}\Big((r_1^2 + r_2^2)\widetilde{d}^2 d + (r_1\sigma_2 + r_2\sigma_1)s\widetilde{d}\Big) - 4\sigma_1\sigma_2 s\widetilde{d}.\end{aligned} \tag{6.39}$$

Next we consider the compatibility condition with respect to τ. By writing the matrix equation (6.31) in entrywise form, with the help of Maple, we obtain, with the prime denoting the derivative with respect to τ,

$$r_1 d' = (r_1^2 + r_2^2)(r_1^2\tau\widetilde{\beta} + r_2^2\tau\widetilde{\beta} + r_1 c\widetilde{\beta} + r_2\widetilde{c}\widetilde{\beta} + r_2 d^2\widetilde{d} - r_1 c^2 d + 2s_1 d + r_2 f) \tag{6.40}$$

$$r_2 d' = (r_1^2 + r_2^2)(r_1^2\tau b + r_2^2\tau b - r_1 cb - r_2\widetilde{c}b - r_1 d^2\widetilde{d} + r_2\widetilde{c}^2 d - 2s_2 d - r_1 f) \tag{6.41}$$

$$r_1 \widetilde{d}' = (r_1^2 + r_2^2)(r_1^2\tau\widetilde{b} + r_2^2\tau\widetilde{b} - r_2\widetilde{c}\widetilde{b} - r_1 c\widetilde{b} - r_2\widetilde{d}^2 d + r_1 c^2\widetilde{d} - 2s_1\widetilde{d} - r_2\widetilde{f}) \tag{6.42}$$

$$r_2 \widetilde{d}' = (r_1^2 + r_2^2)(r_1^2\tau\beta + r_2^2\tau\beta + r_2\widetilde{c}\beta + r_1 c\beta + r_1\widetilde{d}^2 d - r_2\widetilde{c}^2\widetilde{d} + 2s_2\widetilde{d} + r_1\widetilde{f}) \tag{6.43}$$

$$c' = (r_1^2 + r_2^2)(d\beta - \widetilde{d}b) \tag{6.44}$$

$$\widetilde{c}' = (r_1^2 + r_2^2)(\widetilde{d}\widetilde{\beta} - d\widetilde{b}) \tag{6.45}$$

and

$$\begin{aligned}r_1 b' + r_2\widetilde{\beta}' = {} & (r_1^2 + r_2^2)(-r_1 c^2 b + r_2\widetilde{c}^2\widetilde{\beta} - r_1 d\widetilde{d}\widetilde{\beta} + r_2 d\widetilde{d}b + 2s_1 b - 2s_2\widetilde{b} \\ & - r_1^2\tau f - r_2^2\tau f - r_1 cf + r_2\widetilde{c}f) \\ r_2\widetilde{b}' + r_1\beta' = {} & (r_1^2 + r_2^2)(-r_2\widetilde{c}^2\widetilde{b} + r_1 c^2\beta - r_2 d\widetilde{d}\beta + r_1 d\widetilde{d}\widetilde{b} + 2s_2\widetilde{b} - 2s_1 b \\ & - r_2^2\tau\widetilde{f} - r_1^2\tau\widetilde{f} - r_2\widetilde{c}\widetilde{f} + r_1 c\widetilde{f}).\end{aligned}$$

Direct calculations show that all these equations are satisfied by (6.4)–(6.13). □

Lemma 6.7. *Theorem 6.2 holds true if $\tau = 0$.*

Proof. Equations (6.4)–(6.7) follow from [13, Th. 2.4]. The other equations in Theorem 6.2 then follow from (6.32)–(6.35) and (6.41)–(6.42). □

With the help of Lemmas 6.6–6.7, one can prove Theorem 6.2 for $\tau \neq 0$ in the same way as in [14, Sec. 5], where the symmetric case was considered. This is a lengthy and tedious calculation that follows exactly the same plan as in [14]. We note that the same reasoning also yields the solvability statement in Proposition 2.2. We do not go into the details.

Alternative approach to Theorem 6.2

The above reasoning does not give any insight on the origin of the expressions in Theorem 6.2. Therefore, in the remaining part of this section, let us deduce these formulas in a more direct way. The calculations below are partly heuristic in the sense that we will make an ansatz (6.54), (6.62). We start with

Lemma 6.8. *The numbers $d, \widetilde{d}$ in* (6.1) *satisfy the system of coupled second-order differential equations*

$$\begin{aligned}\frac{\partial^2 d}{\partial s^2} &= 4\tau(r_1\sigma_1 - r_2\sigma_2)\frac{\partial d}{\partial s} \\ &\qquad - 4(r_1^2+r_2^2)(\sigma_1^2+\sigma_2^2)\tau^2 d + 8\frac{(r_1\sigma_2+r_2\sigma_1)^2}{r_1 r_2} d^2\widetilde{d} + 8\frac{(r_1\sigma_2+r_2\sigma_1)^3}{r_1r_2(r_1^2+r_2^2)} s d, \quad (6.46)\\ \frac{\partial^2 \widetilde{d}}{\partial s^2} &= -4\tau(r_1\sigma_1 - r_2\sigma_2)\frac{\partial \widetilde{d}}{\partial s} \\ &\qquad - 4(r_1^2+r_2^2)(\sigma_1^2+\sigma_2^2)\tau^2 \widetilde{d} + 8\frac{(r_1\sigma_2+r_2\sigma_1)^2}{r_1 r_2} \widetilde{d}^2 d + 8\frac{(r_1\sigma_2+r_2\sigma_1)^3}{r_1r_2(r_1^2+r_2^2)} s \widetilde{d}. \quad (6.47)\end{aligned}$$

Moreover,

$$\frac{\partial d}{\partial \tau} = -\frac{r_1r_2(r_1^2+r_2^2)}{\sigma_1r_2+\sigma_2r_1}\tau\frac{\partial d}{\partial s} + (r_1^2+r_2^2)^2\frac{\sigma_1r_2-\sigma_2r_1}{\sigma_1r_2+\sigma_2r_1}\tau^2 d + 2(r_1s_1-r_2s_2)d. \quad (6.48)$$

Proof. Equation (6.46) follows from (6.32)–(6.33) and (6.36)–(6.38) after some lengthy algebraic manipulations. Equation (6.47) follows by symmetry. To obtain (6.48), first note that (6.32)–(6.33) imply the relations

$$r_2(\widetilde{c}d - b) - r_1(cd - \widetilde{\beta}) + (r_1^2+r_2^2)\tau d = 0, \quad (6.49)$$

$$r_1r_2\frac{\partial d}{\partial s} = (\sigma_1r_2+\sigma_2r_1)(r_2\widetilde{c}d - r_2b + r_1cd - r_1\widetilde{\beta}) + (r_1^2+r_2^2)(\sigma_1r_2-\sigma_2r_1)\tau d. \quad (6.50)$$

Now by eliminating f from (6.40)–(6.41) we get

$$\frac{\partial d}{\partial \tau} = (r_1^2+r_2^2)(r_2b+r_1\widetilde{\beta})\tau - (r_2b-r_1\widetilde{\beta})(r_1c+r_2\widetilde{c}) - d(r_1^2c^2-r_2^2\widetilde{c}^2) + 2(r_1s_1-r_2s_2)d, \quad (6.51)$$

On account of (6.49) this becomes

$$\frac{\partial d}{\partial \tau} = (r_1^2+r_2^2)(r_2b+r_1\widetilde{\beta}-r_1cd-r_2\widetilde{c}d)\tau + 2(r_1s_1-r_2s_2)d.$$

Combining this with (6.50) we obtain (6.48). □

We seek a solution to the differential equations (6.46)–(6.47) in the form

$$d = e^h g, \qquad \widetilde{d} = e^{-h} g, \quad (6.52)$$

with $h = h(s,\tau)$ an odd function of τ and $g = g(s,\tau)$ an even function of τ (recall Lemma 6.1). Plugging this in (6.46) we find, with again the prime denoting the derivative with respect to s,

$$\begin{aligned}g'' + 2h'g' + ((h')^2 + h'')g &= 4\tau(r_1\sigma_1 - r_2\sigma_2)(g' + h'g) \\ &\qquad - 4(r_1^2+r_2^2)\tau^2(\sigma_1^2+\sigma_2^2)g + 8\frac{(r_1\sigma_2+r_2\sigma_1)^2}{r_1r_2} g^3 + 8\frac{(r_1\sigma_2+r_2\sigma_1)^3}{r_1r_2(r_1^2+r_2^2)} s g. \quad (6.53)\end{aligned}$$

To obtain further progress we make the ansatz

$$\frac{\partial^2 h}{\partial s^2} = 0, \qquad \frac{\partial h}{\partial s} = 2(r_1\sigma_1 - r_2\sigma_2)\tau. \tag{6.54}$$

After a little calculation, (6.53) then simplifies to

$$g'' = -4(r_1\sigma_2 + r_2\sigma_1)^2\tau^2 g + 8\frac{(r_1\sigma_2 + r_2\sigma_1)^2}{r_1 r_2} g^3 + 8\frac{(r_1\sigma_2 + r_2\sigma_1)^3}{r_1 r_2(r_1^2 + r_2^2)} sg. \tag{6.55}$$

We can relate (6.55) to the Painlevé II equation. We have that $q = q(s)$ satisfies $q'' = sq + 2q^3$, if and only if

$$g(s) := c_1 q(c_2 s + c_3) \tag{6.56}$$

satisfies

$$g'' = c_2^2 c_3 g + 2\frac{c_2^2}{c_1^2} g^3 + c_2^3 sg. \tag{6.57}$$

Comparing coefficients with (6.55), we see that

$$c_1 = \frac{(r_1 r_2)^{1/6}}{(r_1^2 + r_2^2)^{1/3}} \tag{6.58}$$

$$c_2 = 2\frac{(r_1\sigma_2 + r_2\sigma_1)}{(r_1 r_2(r_1^2 + r_2^2))^{1/3}} \tag{6.59}$$

$$c_3 = -(r_1 r_2(r_1^2 + r_2^2))^{2/3}\tau^2. \tag{6.60}$$

Finally, substituting the formulas (6.52), (6.56),

$$d = e^h g = e^h \frac{(r_1 r_2)^{1/6}}{(r_1^2 + r_2^2)^{1/3}} q\left(2\frac{(r_1\sigma_2 + r_2\sigma_1)}{(r_1 r_2(r_1^2 + r_2^2))^{1/3}} s - (r_1 r_2(r_1^2 + r_2^2))^{2/3}\tau^2\right) \tag{6.61}$$

in (6.48) and using again (6.54) we find after some calculations,

$$\frac{\partial h}{\partial \tau} = (r_2^4 - r_1^4)\tau^2 + 2(r_1\sigma_1 - r_2\sigma_2)s, \tag{6.62}$$

where we are assuming that the choice of the Painlevé II solution q in (6.61) is independent from τ. From (6.61)–(6.62) and the known result for $\tau = 0$ [13, Th. 2.4] we get the expression for d in Theorem 2.7 (with q the Hastings-McLeod solution to Painlevé II). By symmetry we obtain the expression for $\widetilde{d}$. From (6.44), (6.32), (6.34) and a little calculation we then find

$$\frac{\partial}{\partial \tau} c = -2r_1^{-1}(r_1^2 + r_2^2)\tau C^{-2} q^2(\sigma) = \frac{\partial}{\partial \tau}(-r_1^{-1}C^{-1}u(\sigma)),$$

where the second equality follows from (4.19) and (2.27). Combining this with the known result for $\tau = 0$ [13, Th. 2.4] we get the expression for c in Theorem 2.7. From (6.32) and a little calculation we find the expression for b, while (6.41) yields the formula for f. Finally, the remaining formulas in Theorem 2.7 follow from symmetry considerations, see Lemma 6.1. ☐

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